`Cutting-Planes` by projecting instead of separating

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Plan

1. The general idea of projecting instead of separating
2. Linear robust optimization
3. The Benders reformulation
4. `Column Generation` for graph coloring
5. Conclusions

Input : a polytope P with prohibitively-many constraints

Goal : "upgrade" the standard `Cutting-Planes` (right) to a new method that uses projections inside the polytope P

- The separation sub-problem will be upgraded to the projection sub-problem

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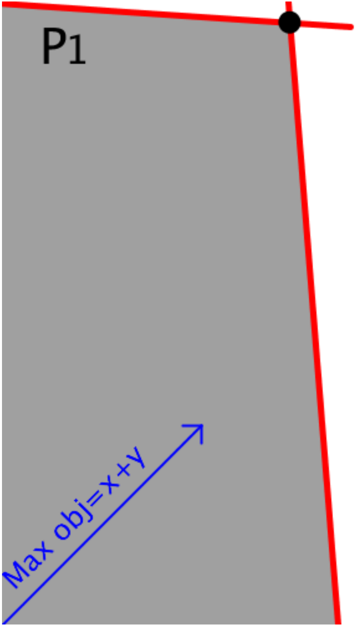
Recall : Each iteration k corresponds to an outer approximation P_k of P and the `Cutting-Planes` has to separate $\texttt{opt}(P_k)$



Given $\mathbf{x} \in P$ and a direction $\mathbf{d} \in \mathbb{R}^n$, the projection of $\mathbf{x}$ along $\mathbf{d}$ asks to find the maximum step length t^* such that $\mathbf{x} + t^*\mathbf{d} \in P$

Using such projections, the new method generates a convergent sequence of inner solutions.

1. At each iteration k, the projection $\mathbf{x}_k \to \mathbf{d}_k$ generates a contact point $\mathbf{x}_k + t_k^*\mathbf{d}_k$ and a first-hit facet

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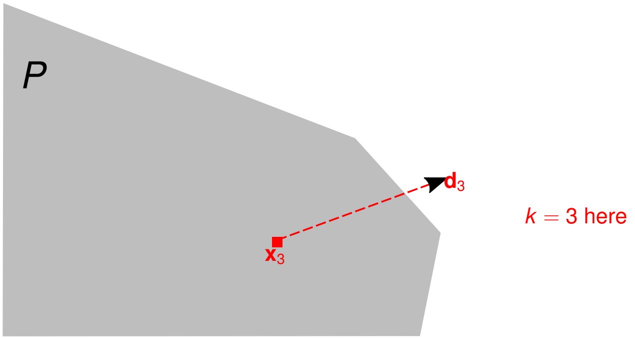

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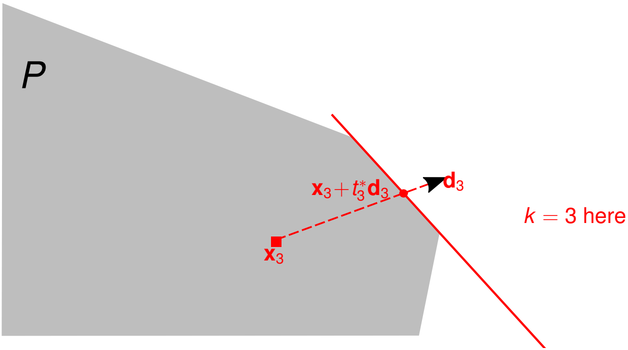
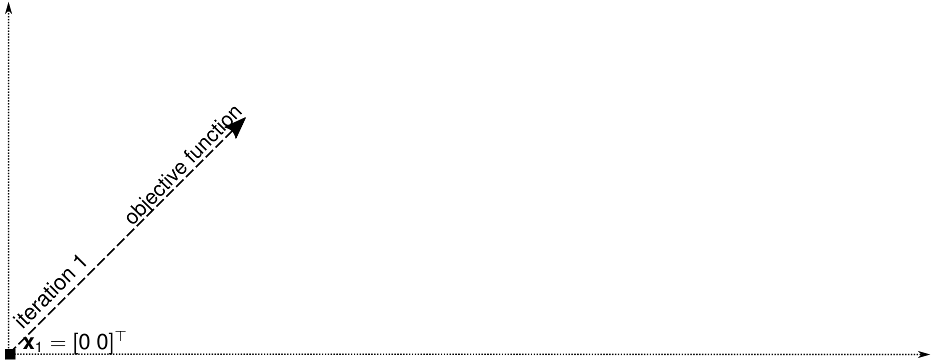


Problems addressed with the new method

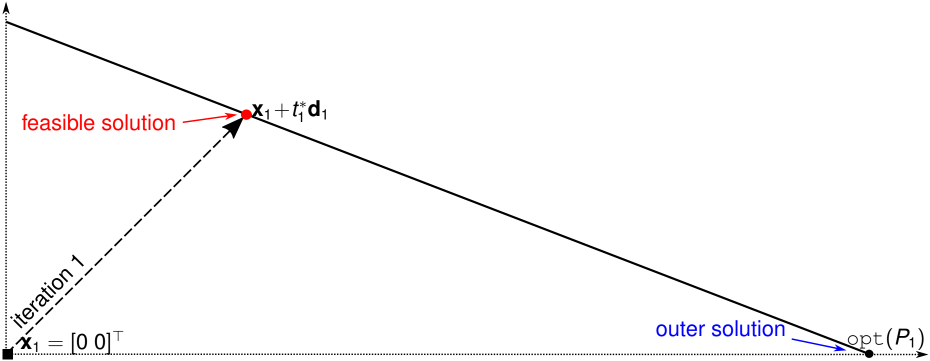
1. Graph coloring (dual polytope in Column Generation)
2. A Benders's `Cutting-Planes` problem (primal polytope)
3. A robust optimization problem (primal polytope)
4. Cutting-Stock with multiple lengths (dual polytope)

Different techniques have been used to solve the projection sub-problem for these different problems :

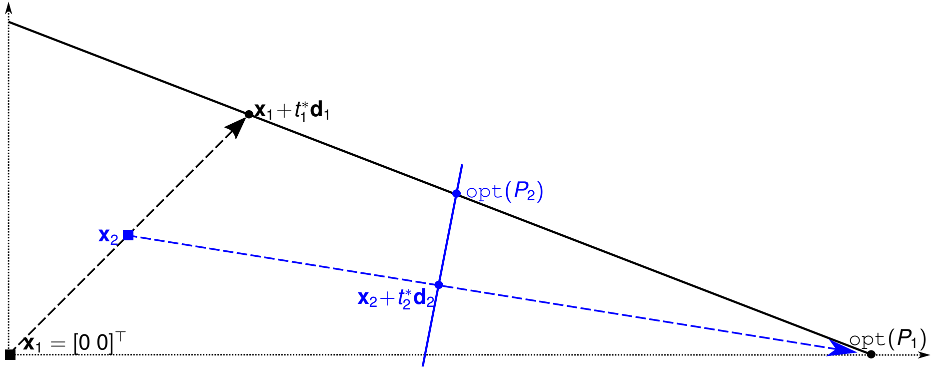
- The Charnes-Cooper transformation for 2, an ad-hoc method for 3 or Dynamic Programming for 4
- I'll focus on 1



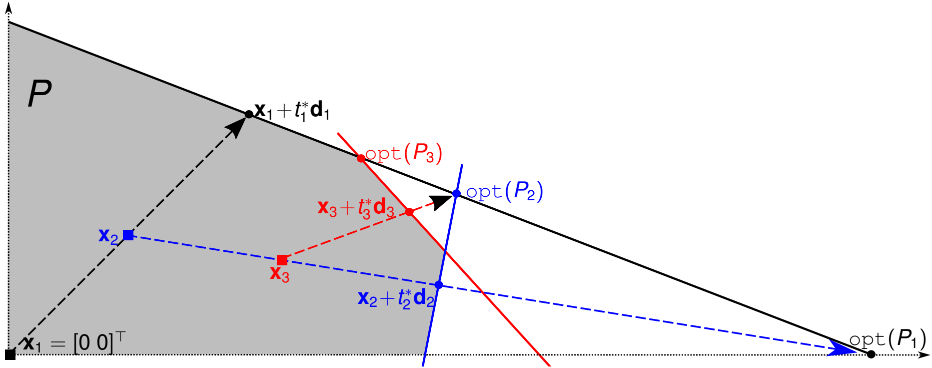
Iteration 1 : uncharted territory, follow objective function, i.e., advance along $\mathbf{x}_1 \rightarrow \mathbf{d}_1$ where $\mathbf{d}_1$ takes the value of the objective function



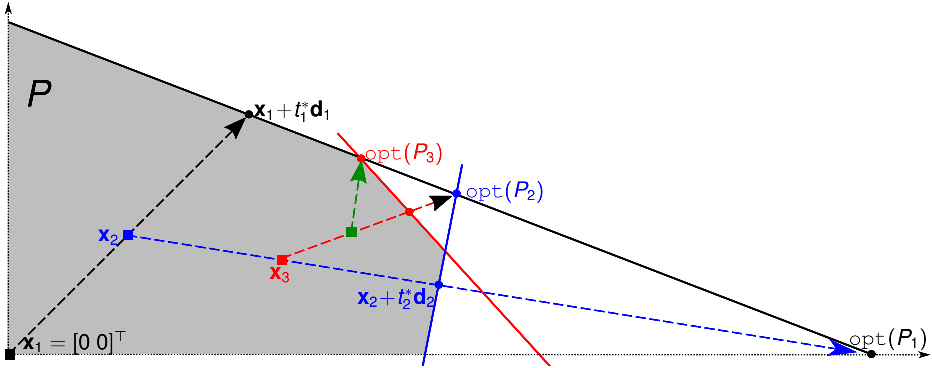
Iteration 1 : found a first outer solution $\texttt{opt}(P_1)$ and a first inner solution (contact point) $\mathbf{x}_1 + t_1^* \mathbf{d}_1$



Iteration 2 : an inner feasible solution (contact point) $\mathbf{x}_2 + t_2^* \mathbf{d}_2$ and a new outer solution. We take $\mathbf{d}_2 = \texttt{opt}(P_1) - \mathbf{x}_2$.



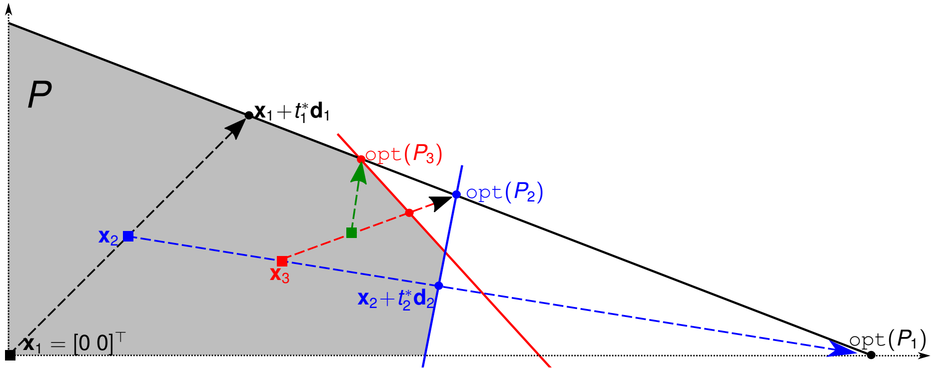
Iteration 3 : the feasible solution $\mathbf{x}_3 + t_3^*\mathbf{d}_3$ is almost optimal



Iteration 4 : optimality of $\texttt{opt}(P_3)$ proved

You can see the proposed method is convergent because it solves a separation problem on $\texttt{opt}(P_k)$ at each iteration k

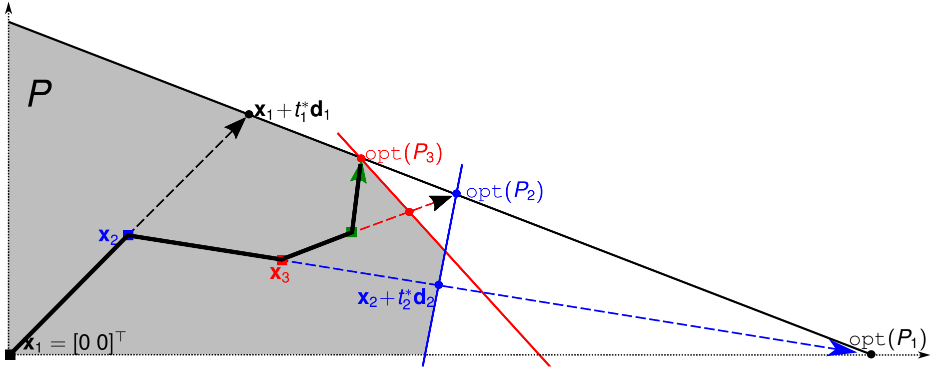
- The convergence proof takes two lines, cool !



Building on existing work [1,2], the new method was deliberately designed to be more general and when possible simpler

[1] Daniel Porumbel. Ray projection for optimizing polytopes with prohibitively many constraints in set-covering column generation. *Mathematical Programming*, 155(1) :147–197, 2016.

[2] Daniel Porumbel. From the separation to the intersection subproblem for optimizing polytopes with prohibitively many constraints in a Benders decomposition context. *Discrete Optimization*, 2018.



Notice the trajectory of the inner points — there is no built-in feature in the `Cutting-Planes` to generate inner points

- each x_k is a point on the last projected segment, i.e., between $\mathbf{x}_{k-1}$ and $\mathbf{x}_{k-1} + t^*_{k-1}\mathbf{d}_{k-1}$
- in this example we choose : $x_k = \mathbf{x}_{k-1} + \frac{1}{2} \cdot t^*_{k-1}\mathbf{d}_{k-1}$

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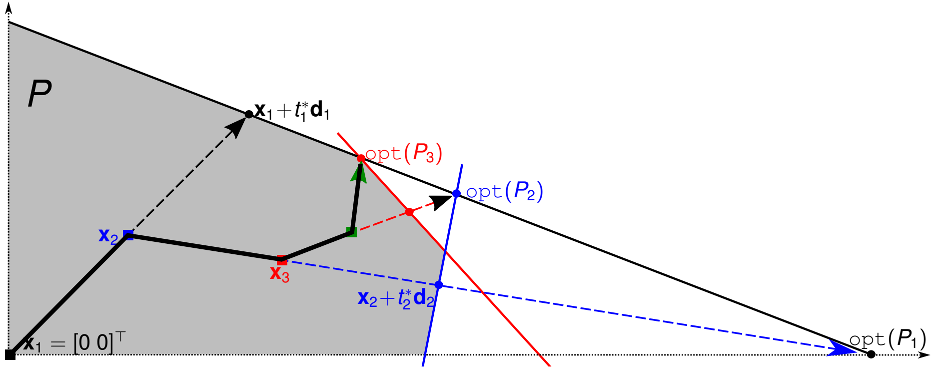
everything was like a movie until here : let's move to real life



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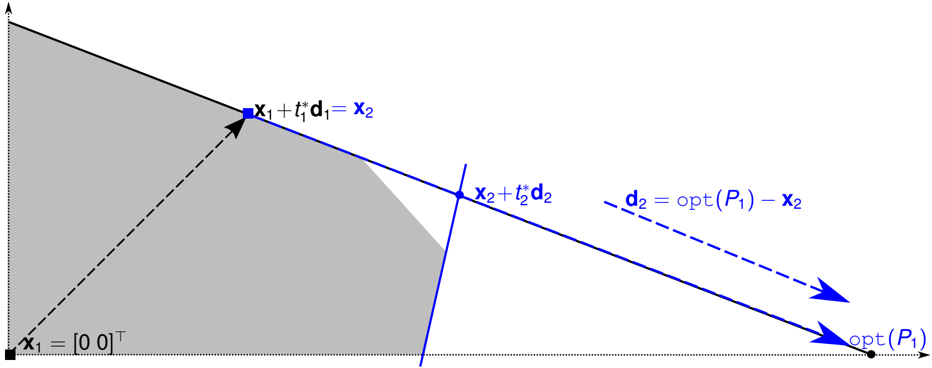

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- we here choose the contact point : $x_k = \mathbf{x}_{k-1} + t^*_{k-1}\mathbf{d}_{k-1}$

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The projection sub-problem

We are given :

$$\max\left\{\mathbf{b}^\top\mathbf{x} : \mathbf{a}^\top\mathbf{x} \le c_a,\ \forall(\mathbf{a}, c_a) \in \texttt{Constr}\right\} = \max\left\{\mathbf{b}^\top\mathbf{x} : \mathbf{x} \in P\right\}$$

Separation sub-problem on $\mathbf{x}$

$$\min\left\{c_a - \mathbf{a}^\top\mathbf{x} : (\mathbf{a}, c_a) \in \texttt{Constr}\right\}$$

Projection sub-problem on $\mathbf{x} \to \mathbf{d}$

$\implies$ find max t^* such that $\mathbf{a}^\top\mathbf{x} + t^* \cdot \mathbf{a}^\top\mathbf{d} \le c_a\ \forall(\mathbf{a}, c_a) \in \texttt{Constr}$

$$t^* = \min\left\{\frac{c_a - \mathbf{a}^\top\mathbf{x}}{\mathbf{a}^\top\mathbf{d}} :\ (\mathbf{a}, c_a) \in \texttt{Constr},\ \mathbf{d}^\top\mathbf{a} > 0\right\}.$$

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Plan

Introducing the Robust Linear Program

Start from a standard LP with a feasible area described by :

$$\mathbf{a}^\top \mathbf{x} \leq c_a \; \forall (\mathbf{a},\ c_a) \in \texttt{Constr}_{\texttt{nom}}$$

For each nominal constraint $(\mathbf{a},\ c_a) \in \texttt{Constr}_{\texttt{nom}}$ one can define a (huge) set of robust constraints

$$(\mathbf{a} + \mathbf{a}')^\top \mathbf{x} \leq c_a,$$

where any $\mathbf{a}'$ belongs to a set a (reasonable) deviation of the nominal coefficients $\mathbf{a}$.

More exactly : $\mathbf{a}' \in \mathbb{R}^n$ is a vector with at maximum Γ non-zero components such that $a'_i \in \{-0.01 \cdot a_i,\ 0,\ 0.01 \cdot a_i\} \; \forall i \in [1..n]$.

- We write $\mathbf{a}' \in \texttt{Dev}_\Gamma(\mathbf{a})$

I only compare to the cutting-planes from [M. Fischetti and M. Monaci. Cutting plane versus compact formulations for uncertain (integer) linear programs. Mathematical Programming Computation, 4(3) :239–273, 2012.]

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$$(\mathbf{a}+\mathbf{a}')^\top(\mathbf{x}+t\mathbf{d}) \le c_a \; \forall(\mathbf{a},\, c_a) \in \mathtt{Constr}_{\mathtt{nom}}, \mathbf{a}' \in \mathtt{Dev}_\Gamma(\mathbf{a}),$$

i.e., we consider all nominal constraints $(\mathbf{a},\, c_a)$ and all their deviations $\mathbf{a}' \in \mathtt{Dev}_\Gamma(\mathbf{a})$

Let's work this formula for each nominal constraint $(\mathbf{a},\, c_a)$

The projection sub-problem asks to minimize

$$\min_{\mathbf{a}' \in \mathtt{Dev}_\Gamma(\mathbf{a})} \frac{c_a - \mathbf{a}^\top\mathbf{x} - \mathbf{a}'^\top\mathbf{x}}{\mathbf{a}^\top\mathbf{d} + \mathbf{a}'^\top\mathbf{d}}$$

This could be written as a linear-fractional program (because $\mathbf{a}'$ can be seen as belonging to a polytope) and solved with the Charness-Cooper transformation.

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A step-by-step t-decreasing algorithm

Start with the t value given by the nominal constraint alone, *i.e.*, fix $\mathbf{a}' = 0$ in formula below :

$$t \leftarrow \frac{c_a - \mathbf{a}^\top \mathbf{x} - \mathbf{a}'^\top \mathbf{x}}{\mathbf{a}^\top \mathbf{d} + \mathbf{a}'^\top \mathbf{d}} \overset{\mathbf{a}'=\mathbf{0}_n}{=} \frac{c_a - \mathbf{a}^\top \mathbf{x}}{\mathbf{a}^\top \mathbf{d}}$$

1. This t is not necessarily optimal because there might exist a different $\mathbf{a}'(\neq \mathbf{0}_n)$ such that

$$t > \frac{c_a - \mathbf{a}^\top \mathbf{x} - \mathbf{a}'^\top \mathbf{x}}{\mathbf{a}^\top \mathbf{d} + \mathbf{a}'^\top \mathbf{d}}$$

equivalent to

$$t \cdot (\mathbf{a}^\top \mathbf{d} + \mathbf{a}'^\top \mathbf{d}) > c_a - \mathbf{a}^\top \mathbf{x} - \mathbf{a}'^\top \mathbf{x}$$

2. Solve $\min_{\mathbf{a}' \in \mathtt{Dev}_\Gamma(\mathbf{a})} c_a - \mathbf{a}^\top \mathbf{x} - \mathbf{a}'^\top \mathbf{x} - t \cdot (\mathbf{a}^\top \mathbf{d} + \mathbf{a}'^\top \mathbf{d})$

3. Repeat from 1 while the optimum of above LP is below 0

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A step-by-step t-decreasing algorithm

The resulting t is given as input to the next constraint of $\texttt{Constr}_{\texttt{nom}}$, to iteratively apply steps **1** — **3** to all $\texttt{Constr}_{\texttt{nom}}$

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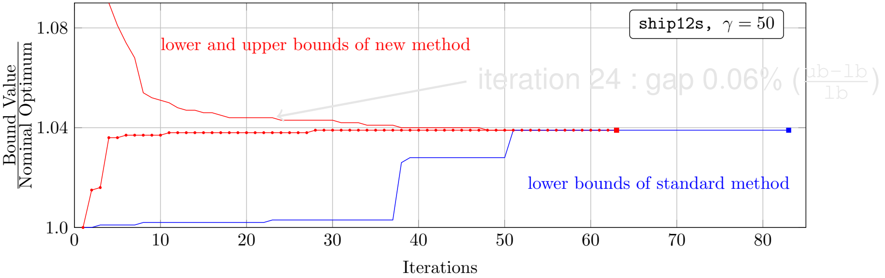
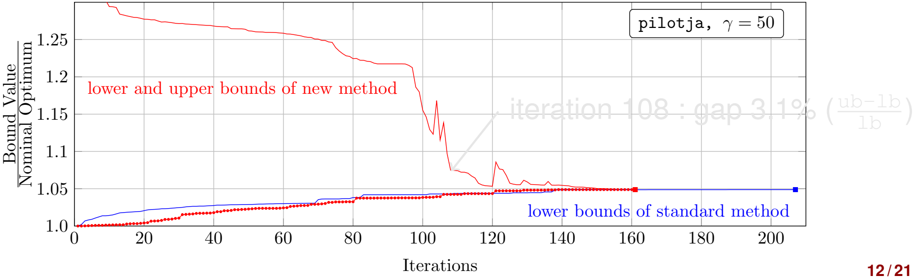
3 Repeat from **1** while the optimum of above LP is below 0

Results on the robust linear program

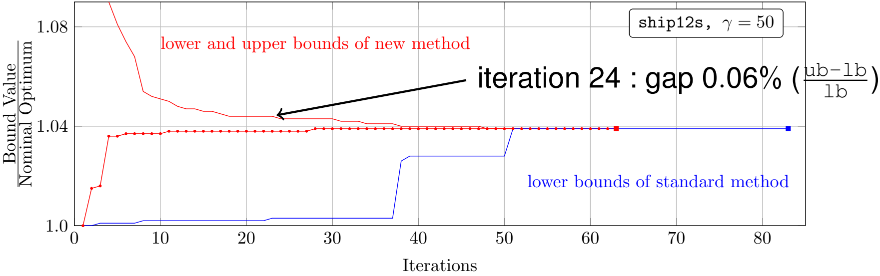
Each interior point is defined as : $\mathbf{x}_k = \mathbf{x}_{k-1} + \frac{1}{10} \cdot t^*_{k-1}\mathbf{d}_{k-1}$

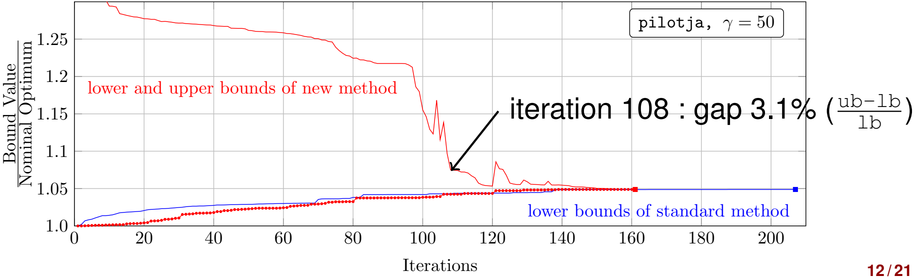




Results on the robust linear program

Each interior point is defined as : $\mathbf{x}_k = \mathbf{x}_{k-1} + \frac{1}{10} \cdot t^*_{k-1}\mathbf{d}_{k-1}$





Plan

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Du modèle de départ aux reformulations

$$\min \mathbf{d}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{y}$$
$$\mathbf{D}\mathbf{x} \geq \mathbf{e}$$
$$\mathbf{B}\mathbf{x} + \mathbf{A}\mathbf{y} \geq \mathbf{c}$$
$$\mathbf{x} \in \mathbb{Z}^n_+, \ \mathbf{y} \geq \mathbf{0}$$

coût des flux, on va utiliser $\mathbf{b} = \mathbf{0}$

Contraintes de design

Le flux $\mathbf{y}$ doit pouvoir passer

x est un nombre d'unités à faire fonctionner.

- câbles à monter
- entrepôts à ouvrir

y est un coût des flux qui passent, coût d'affectations, etc

Du modèle de départ aux reformulations

$$\min \mathbf{d}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{y}$$
$$\mathbf{D}\mathbf{x} \geq \mathbf{e}$$
$$\mathbf{B}\mathbf{x} + \mathbf{A}\mathbf{y} \geq \mathbf{c}$$
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coût des flux, on va utiliser $\mathbf{b} = \mathbf{0}$

Contraintes de design

Le flux $\mathbf{y}$ doit pouvoir passer

Reformulation 1 :

$$\min \mathbf{d}^\top \mathbf{x} + \hat{z}$$
$$\mathbf{D}\mathbf{x} \geq \mathbf{e}$$
$$\hat{z} = \min \left\{ \mathbf{b}^\top \mathbf{y} : \mathbf{B}\mathbf{x} + \mathbf{A}\mathbf{y} \geq \mathbf{c}, \mathbf{y} \geq \mathbf{0} \right\}$$
$$\mathbf{x} \in \mathbb{Z}_+^n$$

Du modèle de départ aux reformulations

$$\begin{aligned} \min \mathbf{d}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{y} \\ \mathbf{D}\mathbf{x} \geq \mathbf{e} \\ \mathbf{B}\mathbf{x} + \mathbf{A}\mathbf{y} \geq \mathbf{c} \\ \mathbf{x} \in \mathbb{Z}^n_+, \ \mathbf{y} \geq \mathbf{0} \end{aligned}$$

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On va dualiser ce PL

coût des flux, on va utiliser $\mathbf{b} = \mathbf{0}$ $\Longrightarrow \ \hat{z} = 0$

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Reformulation 2 :

$$\begin{aligned} &\min \mathbf{d}^\top \mathbf{x} + \hat{z} \\ &\mathbf{D}\mathbf{x} \geq \mathbf{e} \\ &\hat{z} = \max\{(\mathbf{c} - \mathbf{B}\mathbf{x})^\top \mathbf{u} : \ \mathbf{u} \in \mathscr{P}\}, \\ &\mathbf{x} \in \mathbb{Z}_+^n \end{aligned}$$

$$\boxed{\mathscr{P} = \{\mathbf{u} \geq \mathbf{0} : \mathbf{A}^\top \mathbf{u} \leq \mathbf{b}\}}$$

coût des flux, on va utiliser $\mathbf{b} = \mathbf{0}$ $\implies \hat{z} = 0$

$$\min \mathbf{d}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{y}$$
$$\mathbf{D}\mathbf{x} \geq \mathbf{e}$$
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$$\mathbf{x} \in \mathbb{Z}_+^n, \ \mathbf{y} \geq \mathbf{0}$$

Reformulation 3 :

$$\min \mathbf{d}^\top \mathbf{x} + \hat{z}$$
$$\mathbf{D}\mathbf{x} \geq \mathbf{e}$$
$$\hat{z} = max\{(\mathbf{c} - \mathbf{B}\mathbf{x})^\top \mathbf{u} : \mathbf{u} \in \mathscr{P}\}$$
$$0 \geq (\mathbf{c} - \mathbf{B}\mathbf{x})^\top \mathbf{u} \quad \forall \mathbf{u} \in \mathscr{P},$$
$$\mathbf{x} \in \mathbb{Z}_+^n$$

$$\mathscr{P} = \{\mathbf{u} \geq \mathbf{0} : \mathbf{A}^\top \mathbf{u} \leq \mathbf{0}\}$$

Du modèle de départ aux reformulations

$$\min \mathbf{d}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{y}$$
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Reformulation 4 :

$$\min \mathbf{d}^\top \mathbf{x}$$
$$\mathbf{D}\mathbf{x} \geq \mathbf{e}$$
$$(\mathbf{B}\mathbf{x})^\top \mathbf{u} \geq \mathbf{c}^\top \mathbf{u} \quad \forall \mathbf{u} \in \mathscr{P},$$
$$\mathbf{x} \in \mathbb{Z}_+^n$$

Les coupes Benders sont définies par les rayons **u** du polytope Benders $\mathscr{P}$

$$\mathscr{P} = \{\mathbf{u} \geq \mathbf{0} : \mathbf{A}^\top \mathbf{u} \leq \mathbf{0}\}$$

Du modèle de départ aux reformulations

$$\min \mathbf{d}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{y}$$
$$\mathbf{D}\mathbf{x} \geq \mathbf{e}$$
$$\mathbf{B}\mathbf{x} + \mathbf{A}\mathbf{y} \geq \mathbf{c}$$

Projection $\mathbf{x} \to \mathbf{d}$

$$\max_{\mathbf{u} \in \mathscr{P}} \left\{ \frac{(\mathbf{B}\mathbf{x})^\top \mathbf{u} - \mathbf{c}^\top \mathbf{u}}{-(\mathbf{B}\mathbf{d})^\top \mathbf{u}} : -(\mathbf{B}\mathbf{d})^\top \mathbf{u} > 0 \right\}$$

Reformulation 4 :

$$\min \mathbf{d}^\top \mathbf{x}$$
$$\mathbf{D}\mathbf{x} \geq \mathbf{e}$$
$$(\mathbf{B}\mathbf{x})^\top \mathbf{u} \geq \mathbf{c}^\top \mathbf{u} \quad \forall \mathbf{u} \in \mathscr{P},$$
$$\mathbf{x} \in \mathbb{Z}_+^n$$

$$\mathscr{P} = \{\mathbf{u} \geq \mathbf{0} : \mathbf{A}^\top \mathbf{u} \leq \mathbf{0}\}$$

Résolution sous-problème d'intersection

Il faut résoudre

$$t^* = \max_{\mathbf{u}\in\mathscr{P}} \left\{ \frac{(\mathbf{Bx})^\top \mathbf{u} - \mathbf{c}^\top \mathbf{u}}{-(\mathbf{Bd})^\top \mathbf{u}} \; : \; -(\mathbf{Bd})^\top \mathbf{u} > 0 \right\}$$

$$\texttt{avec} \quad \mathscr{P} = \left\{ \mathbf{u} \geq \mathbf{0} : \mathbf{A}^\top \mathbf{u} \leq \mathbf{0} \right\}$$

Charnes–Cooper transformation :

$$\bar{\mathbf{u}} = \frac{\mathbf{u}}{-(\mathbf{Bd})^\top \mathbf{u}}$$

Using $\mathbf{u} \in \mathscr{P} \implies \mathbf{A}^\top \bar{\mathbf{u}} \leq \mathbf{0}, \bar{\mathbf{u}} \geq \mathbf{0}, -(\mathbf{Bd})^\top \bar{\mathbf{u}} = 1$, we obtain

$$t^* = \min \left\{ (\mathbf{Bx})^\top \bar{\mathbf{u}} - \mathbf{c}^\top \bar{\mathbf{u}} : \; \mathbf{A}^\top \bar{\mathbf{u}} \leq \mathbf{0}, \; \bar{\mathbf{u}} \geq \mathbf{0}, \; -(\mathbf{Bd})^\top \bar{\mathbf{u}} = 1 \right\}$$

Résultats

Instance		Projective Cutting-Planes							Standard Cutting-Planes				
	OPT	Best IP Sol	Iterations avg	(dev)	Time [secs] avg	(dev)	Time solve master		Iterations avg	(dev)	Time [secs] avg	(dev)	Time solve master
a	42.333	48	22.8	(1)	0.06	(0.002)	4.4%		35	(4.9)	0.09	(0.01)	5.5%
b	245.67	265	73.8	(2.7)	0.2	(0.006)	6.1%		131	(11.8)	0.4	(0.04)	8.7%
c	204.33	220	56.5	(1.5)	0.2	(0.004)	4.9%		78.5	(16)	0.2	(0.05)	5.8%
d	299.33	317	67.5	(3)	0.2	(0.01)	4.3%		104	(4.3)	0.4	(0.02)	6.1%
e	67.333	77	35.4	(0.8)	0.1	(0.006)	4.2%		39.5	(5.5)	0.1	(0.02)	5.5%
a	46		174	(27.4)	7.4	(5.8)	89.5%		229	(146)	9.6	(3)	95%
b	260		824	(206)	1073	(636)	99.5%		2987	(2427)	4129	(819)	99.8%
c	214		242	(27.1)	99	(31.6)	98.4%		526	(442)	378	(70.8)	99.6%
d	313		336	(53.4)	321	(103)	99.2%		1315	(1049)	2367	(469)	99.8%
e	74		1336	(138)	4907	(1640)	99.8%		2250	(1292)	6703	(2857)	99.9%

Les 5 derniers lignes concernent le modèle IP.
« dév » = déviation standard

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We need to focus on the dual. The primal master is :

$$
\begin{array}{rll}
 & \min \sum c_a y_a & \\
\mathbf{x}: & \sum a_i y_a \quad \geq b_i & \forall i \in [1..n] \\
 & y_a \geq 0 & \forall (\mathbf{a}, c_a) \in \texttt{Constr}
\end{array}
$$

The dual LP is :

$$
P \begin{cases}
 & \max \mathbf{b}^\top \mathbf{x} \quad (= \max \mathbf{1}_n^\top \mathbf{x}) \\
y_a : & \mathbf{a}^\top \mathbf{x} \leq c_a, \quad \forall (\mathbf{a}, c_a) \in \texttt{Constr} \\
 & \mathbf{x} \geq \mathbf{0}_n
\end{cases}
$$

- $c_a = 1$ for each stable **a** (each color counts once)

We re-write the graph coloring problem :

$$P \begin{cases} \max \mathbf{1}_n^\top \mathbf{x} \\ \mathbf{a}^\top \mathbf{x} \leq 1, \quad \text{for any stable } \mathbf{a} \in \{0,1\}^n \\ \mathbf{x} \geq \mathbf{0}_n \end{cases}$$

The projection sub-problem on $\mathbf{x} \to \mathbf{d}$

$$t^* = \min \left\{ \frac{1 - \mathbf{a}^\top \mathbf{x}}{\mathbf{a}^\top \mathbf{d}} : \ \mathbf{a} \in \texttt{STAB}, \ \mathbf{d}^\top \mathbf{a} > 0 \right\},$$

where STAB is the set of stables that can be written as

$$\begin{aligned} a_i + a_j \leq 1 \ \forall \{i,j\} \in E \\ a_i \in \{0,1\} \qquad \forall i \in V \end{aligned}$$

$$t^* = \min_{\mathbf{a}} \frac{1 - \mathbf{x}^\top \mathbf{a}}{\mathbf{d}^\top \mathbf{a}}$$

$$\mathbf{d}^\top \mathbf{a} > 0$$

$$\texttt{STAB} \begin{cases} a_i + a_j \leq 1, & \forall \{i, j\} \in E \\ a_i \in \{0, 1\} & \forall i \in [1..n] \end{cases}$$

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$$t^* = \min_{\mathbf{a},\overline{\alpha}} \overline{\alpha} - \mathbf{x}^\top \overline{\mathbf{a}}$$

$$\overline{a}_i + \overline{a}_j \leq \overline{\alpha} \quad \forall \{i,j\} \in E$$

$$\mathbf{d}^\top \overline{\mathbf{a}} = 1$$

$$\overline{a}_i \in \{0, \overline{\alpha}\} \quad \forall i \in [1..n]$$

$$\overline{\alpha} \geq 0$$

$$\Longleftrightarrow$$

$$\overline{\mathbf{a}} = \frac{\mathbf{a}}{\mathbf{d}^\top \mathbf{a}} \text{ and } \overline{\alpha} = \frac{1}{\mathbf{d}^\top \mathbf{a}}$$

This is a discrete Charnes–Cooper transformation !

The separation sub-problem is :

$$\min 1 - \mathbf{x}^\top \mathbf{a}$$

$$a_i + a_j \leq 1, \quad \forall \{i,j\} \in E$$

$$a_i \in \{0,1\} \quad \forall i \in [1..n]$$

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$$\boxed{a_i \in \{0,1\}} \quad \forall i \in [1..n]$$

The continuity-breaking constrains remain similar.

$$t^* = \min_{\mathbf{a}} \frac{1 - \mathbf{x}^\top \mathbf{a}}{\mathbf{d}^\top \mathbf{a}}$$

$$\mathbf{d}^\top \mathbf{a} > 0$$

$$\texttt{STAB} \begin{cases} a_i + a_j \leq 1, & \forall \{i,j\} \in E \\ a_i \in \{0,1\} & \forall i \in [1..n] \end{cases}$$

$$t^* = \min_{\mathbf{a},\overline{\alpha}} \overline{\alpha} - \mathbf{x}^\top \overline{\mathbf{a}}$$

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$$\mathbf{d}^\top \overline{\mathbf{a}} = 1$$

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$$\overline{\alpha} \geq 0$$

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This is a discrete Charnes–Cooper transformation!

The separation sub-problem is :

$$\min 1 - \mathbf{x}^\top \mathbf{a}$$

$$\boxed{a_i + a_j \leq 1, \quad \forall \{i,j\} \in E}$$

$$a_i \in \{0,1\} \qquad \forall i \in [1..n]$$

The edge inequalities could have been replaced by

any others imaginable

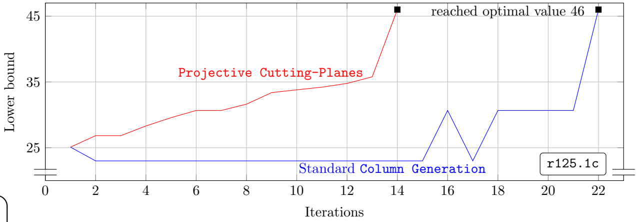
Each interior point $\mathbf{x}_k$ yields a lower bound

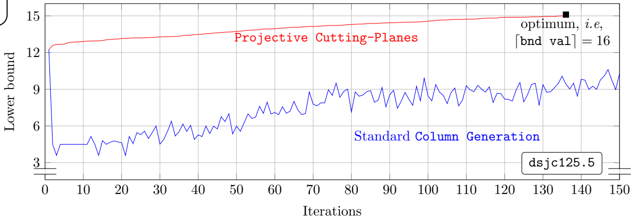
We used

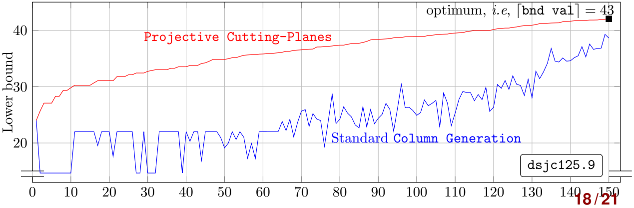
$$\mathbf{x}_k = \mathbf{x}_{k-1} + t^*_{k-1}\mathbf{d}_{k-1}$$

so that $\mathbf{x}_k$ = the last contact point

For Column Generation, we used Lagrangian bounds







Further results on graph coloring

Each first row in black : the standard method

Each second row in red : the new method.

Three lower bounds found along the search

instance	beginning iter:lb/time	mid iteration iter:lb/time	last iteration iter:lb/time
r125.1	34 :2.35/0.06	36 :2.62/0.06	47 :5/0.08
	5 :2.35/0.18	17 :2.61/0.68	20 :5/0.80
dsjc125.5	253 :13.08/365	306 :14.27/634	378 :15.08/1101
	16 :13.04/213	62 :14.001/1288	136 :15.003/4077
dsjc125.9	70 :25.67/24.4	134 :34.13/78	171 :42.11/136
	2 :25.67/7.3	44 :34.11/109	150 :42.03/486

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We proposed `Projective Cutting-Planes` :

1. The driving force is a sequence of inner solutions (a bit like in IPM) that are not available by default in standard `Cutting-Planes`
   - If one wants to calculate inner solutions during a standard `Cutting-Planes`, it may be possible but : 1) one has to apply some ad-hoc method and 2) such inner solutions will always remain a by-product of the algorithm
2. The lower bounds of the new method are monotonically increasing over the iterations : the infamous "yo-yo" effect is finished

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• The new method discovered a new lower bound for a very well studied graph `dsjc250.1`
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