Revisiting the bijection between planar maps and well labeled trees

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Abstract

The bijection between planar graphs and *well labeled trees* was published by Cori and Vauquelin in 1981 [5]. It was afterwords used several times to generate random planar graphs [6]; certain ideas were used more recently to propose other combinatorial bijections [1, 3, 7, 8]. In this short note, we review in a more intuitive manner the initial proof and we also provide alternative arguments for an essential theorem: "the splitting of a map is a map".

1 Introduction

A map is an embedding of a connected graph into the 2-dimensional sphere. The enumerative theory of maps has a long history in combinatorics and it was originally motivated by the four color theorem [2]. This work began with the pioneering articles of Tutte who provided enumerations for several families of maps in his "census" papers in the 60's [9–12]. Since then, the theory has made important progress and numerous other enumerations and bijections have been studied [1, 3, 4, 7, 8]. A particularly original bijection is the one between planar maps and well labeled trees; it was found by Cori and Vauquelin [5] in 1981 and it provided inspiration for most of the more recent bijections.

This one-to-one correspondence between planar maps and well labeled trees is proved in the following manner: starting with a planar map, Cori and Vauquelin [5] construct a new map — the *splitting* of the initial map, which they prove to be a tree with specific properties. A critical point of the construction is to show that the splitting of a map is also a map (theorem 2.3 in [5], page 1029). After that, the main result is a consequence of a series of propositions which follow by checking other map properties such as the genus or the number of cycles. We present in this article a completed proof of this theorem (there was a slight incorrection in the initial version) as well as alternative proofs for most other propositions.

To make the paper self-contained, we start by defining the basic general notions. We use the same notations as Cori and Vauquelin [5] in order to make the analogy easier. The main theorem, with both the initial and alternative proof, is presented in section 2. Some concluding remarks follow in the last section.

Let us denote by Z_m the set of non zero integers whose absolute value is at most m. A permutation acting on Z_m is a bijective function $\sigma : Z_m \to Z_m$. Very often it is useful to denote it in the cycle form. For example $\{(1,-1,2,-3),(3,-2)\}$ represents a permutation $\sigma : Z_3 \to Z_3$ composed of two cycles: $1 \xrightarrow{\sigma} -1 \xrightarrow{\sigma} 2 \xrightarrow{\sigma} -3 \xrightarrow{\sigma} 1$ and $3 \xrightarrow{\sigma} -2 \xrightarrow{\sigma} 3$. The *conjugate* of permutation σ acting on Z_m is the permutation $\bar{\sigma}$ such that $\bar{\sigma}(a) = \sigma(-a)$ $\forall a \in Z_m$.

We say that σ and $\bar{\sigma}$ generate a transitive group on Z_m if for every $a, b \in Z_m$ there exists integers $i_1, i_2, \ldots i_n$ such that $\sigma^{i_1}\bar{\sigma}^{i_2}\sigma^{i_3}\ldots\sigma^{i_{n-1}}\bar{\sigma}^{i_n} = b$. For any two such a and b we say that a and b are linked in σ. A map is a permutation σ acting on Z_m such that σ and $\bar{\sigma}$ generate a transitive group on Z_m. This transitivity is the combinatorial correspondent of the map connectivity. In a more intuitive approach, we can say that σ encodes vertices and $\bar{\sigma}$ encodes edges (see the map in fig. 1).

Let σ be a permutation acting on Z_m and B be a subset of Z_m. We define the closure $\sigma^*(B)$ by: $\sigma^*(B) = \{a \in A \ : \ \exists n \in N, \exists b \in B \ s.\ t.\ a = \sigma^n(b)\}$ If $\sigma^*(B) = B$, then B is *saturated* by σ. The restriction of σ to B is the permutation $\sigma_{/B}$ obtained from σ by erasing the elements of $Z_m - B$ from all the cycles of σ. Then, $\sigma_{/B}(x)$ is the first element of $\sigma(x), \sigma^2(x), \sigma^3(x), \ldots$ belonging to B.

After defining these notions, the first step toward the well labeled tree is to construct a sequence of subsets $B_0, B_2, \ldots, B_p$ in the following manner: $B_0 = \bar{\sigma}^* \{1\}$ and for any $i \geq 0$:

$$B_{2i+1} = \sigma^* B_{2i} - B_{2i} = \{b \in \sigma^* B_{2i} : b \notin B_{2i}\}$$
$$B_{2i} = \bar{\sigma}^* B_{2i-1} - B_{2i-1} = \{b \in \bar{\sigma}^* B_{2i-1} | b \notin B_{2i-1}\}$$

We show in Figure 1 an intuitive representation of the way a map splits into disjoint B_i's. Given the permutation σ as in this example, one could automatically compute $\bar{\sigma}, B_1, B_2, B_3, B_4$:

- $\bar{\sigma} = \{(1,2,3,4-9)(9)(-4,5,6,7,8,-5,-3,-2,-1)(8,10,-10,-7,-6)\}$
- B_0 ={1,2,3,4-9}, B_1 ={9,-4,5,-3,-1,-1}, B_2 ={6,7,8,-5}, B_3 ={-8,-7,-6,10}, B_4 ={-10}

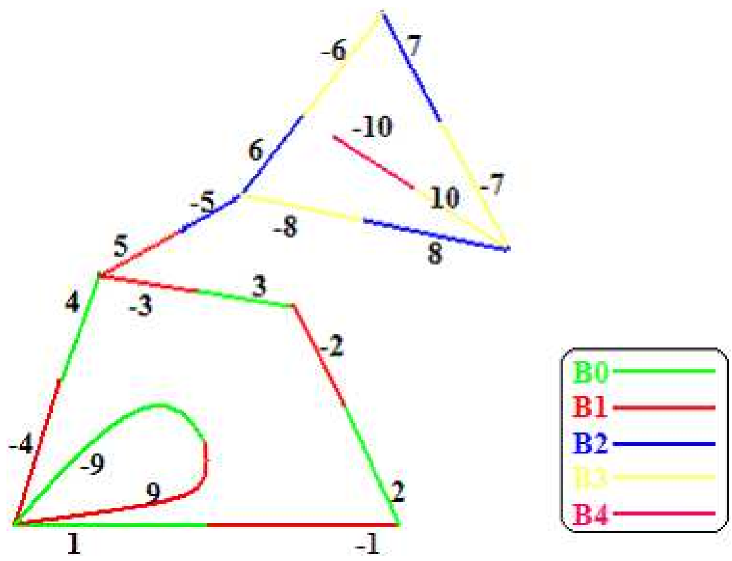

Figure 1: The splitting of $\sigma = \{\ (1,-4,-9,9),\ (-1,2),\ (-2,3),\ (-3,4,5),\ (-5,6,8),\ (-6,7),\ (-7,8,10),\ (-10)\ \}$.

Now we can state the formal definition of the splitting of a map:

$$\hat{\sigma}(a) = \begin{cases} \sigma_{/B_{2i}}(a), & if\ a \in B_{2i} \\ \bar{\sigma}_{/B_{2i+1}}(a), & if\ a \in B_{2i+1} \end{cases}$$

For the above example σ, we thus obtain the labeled tree $\hat{\sigma} = \{(1,-9), (2), (3), (4), (9), (-1,-4,5,-3,-2), (6,-5), (7), (8), (-8,10,-7,-6), (-10)\}$ from Figure 2.

1.1 Useful Lemmas

In order to deal with the main theorem, let us prove three important properties of the set $\{B_0, B_2, \ldots, B_p\}$. The first two correspond to properties 2.1 and 2.2 in the original proof [5, pages 1028–1029]; we also provide alternative proofs for the non-trivial facts.

Lemma 1 *The permutation σ saturates $B_{2i} \bigcup B_{2i+1}$ and $\bar{\sigma}$ saturates $B_{2i+1} \bigcup B_{2i+2}$.*

Proof This is a direct consequence of the definition since we define $B_{2i+1} = \sigma^* B_{2i} - B_{2i}$ and thus $B_{2i} \bigcup B_{2i+1} = \sigma^* B_{2i}$. The second part can be treated similarly by exchanging σ and $\bar{\sigma}$.

Lemma 2 *The set $\{B_1, B_2, \ldots B_p\}$ is a partition of Z_m.*

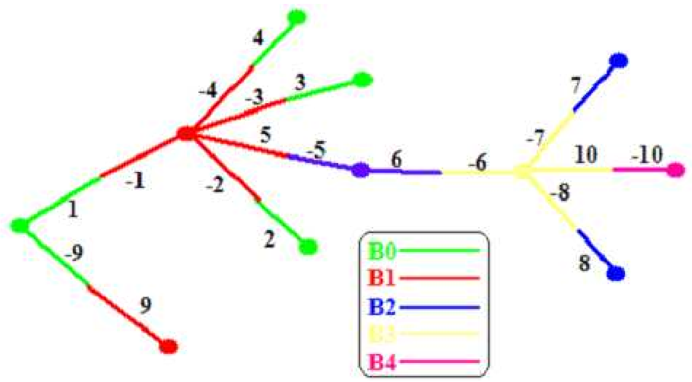

Figure 2: The labeled map $\hat{\sigma}$ associated with σ from our example in fig. 1. It is a tree because the σ is planar.

Proof We suppose that the above set is not a partition. Let B_i and B_j (with $i < j$) be two sets containing a common element a such that j is minimum with the property that $B_1, B_2, \ldots B_{j-1}$ are disjoint. We can consider $j = 2k + 1$ as the opposite case $j = 2k$ can be treated similarly by exchanging σ and $\bar{\sigma}$. From the definition of B_j, we know that there exists $b \in B_{j-1}$ such that a and b are in the same cycle of σ. From the property 1 we also know that σ saturates either $B_i \bigcup B_{i+1}$ or $B_i \bigcup B_{i-1}$ and thus b is also an element of $B_{i-1} \bigcup B_i \bigcup B_{i+1}$. Afterwords, from the definition we know that any two consecutive B_i's are disjoint and thus $i+1$ is at most $j-1$. Practically this means that b appears twice in $B_1, B_2, \ldots B_{j-1}$ (in $B_{i-1} \bigcup B_i \bigcup B_{i+1}$ and also in B_{j-1}). This contradicts the initially stated fact that the sets $B_1, B_2, \ldots B_{j-1}$ are disjoint.

Lemma 3 *If $x \in B_j$, then $-x \in B_{j-1} \cup B_j \cup B_{j+1}$.*

Proof We consider $j = 2k$, the case $j = 2k+1$ can be treated similarly by exchanging σ and $\bar{\sigma}$. Let us assume that $x \in B_{2k}$ and $-x \in B_i$ such that $i \leq 2k - 2$. From lemma 1, $\sigma(x) \in B_{2k} \cup B_{2k+1}$; moreover $\bar{\sigma}(-x)$ belongs to $B_{2k-3} \cup B_{2k-2}$ if $i \in \{2k-2,\ 2k-3\}$ or to other sets which do not intersect $B_{2k} \cup B_{2k+1}$ if $i \leq 2k - 4$. This is a contradiction because $\bar{\sigma}(-x) = \sigma(x)$.

2 Main Theorem and the Completed Proof

The following theorem corresponds to theorem 2.3. in the paper of Cori and Vauquelin [5, page 1029].

Theorem 4 *The splitting of a map is a map.*

Cori and Vauquelin's proof Since $\hat{\sigma}$ is bijective on each B_i, it is also bijective on their union Z_m. In order to prove the theorem, we have to prove that $\hat{\sigma}$ and $\bar{\hat{\sigma}}$ generate a transitive group on Z_m. First, we shall construct for each a and b satisfying $a = \sigma(b)$ or $a = \bar{\sigma}(a)$ a word f of the alphabet $\{\hat{\sigma}, \bar{\hat{\sigma}}\}$ such that $a = f(b)$. The final result will then follows from the transitivity of $\bar{\sigma}$ and σ.

To show the existence of the word f for any $a = \sigma(b)$ and b, they first treat the case where $a \in B_n$ such that $B_{n+1} = \emptyset$ (see [5], theorem 2.3 at page 1029). This proof[1] is the base case of the induction reasoning they perform later for the general case in which $B_{n+1} \neq \emptyset$. The induction hypothesis is that the word f exists for all $a = \sigma(b)$ and b such that a belongs to a set B_j with $j > i = 2k$. Their slight incorrection is in the proof of the induction step when they state that there exists f_l such that $b_{l+1} = f_l(b)$ for all l such that $1 \leq l \leq p-1$ (page 1030, "By induction there exists f_l such that…"). This statement is false for $l = p-1$ because the equation is equivalent to $b_p = f_{p-1}(b)$. The induction hypothesis assures the existence of a word f such that $a = f(b)$ only if $a = \sigma(b) \in B_j$ (with $j \geq 2k+1$). This is not the case for $a = b_p$ ($b_p \notin B_j$ with $j \geq 2k+1$) and b and one cannot use the induction to state $a = b_p = f(b)$.

The completed proof Let us first show the existence of the word f of the alphabet $\{\sigma, \hat{\sigma}\}$ such that $-x = f(x)$ for all x. We start from $\hat{\sigma}^n(\bar{\hat{\sigma}}(x)) = \hat{\sigma}^{n+1}(-x)$ $\forall n \geq 1$; moreover there always exists an i such that $\sigma^i(-x) = -x$, thus $\hat{\sigma}^{i-1}(\bar{\hat{\sigma}}(x)) = \hat{\sigma}^{i-1+1}(-x) = -x$.

We show the existence of the word f such that $y = f(x)$ for any x and y such that $y = \sigma(x)$ (the cases $y = \bar{\sigma}(x)$ can be treated similarly by exchanging σ and $\bar{\sigma}$). From now on, we say x links to y if and only if there is a word f of the $\{\hat{\sigma}, \sigma\}$ alphabet such that $y = f(x)$. Unlike in the induction proof, we prefer to present an new original method that directly analyzes one by one the four possible cases of x's and y's. We show that x links to y in the following situations:

(a) $x \in B_{2k}$ and $\sigma(x) = y \in B_{2k}$

In this case, the existence of the word f is is a direct consequence of the definition of $\hat{\sigma}$ over B_{2k}. Since $\hat{\sigma}(x)$ is $\sigma(x)$ when $x, \sigma(x) \in B_{2k}$, then $y = \hat{\sigma}(x)$ and thus $f = \hat{\sigma}$.

[1] The proof is correct but it only has one incomplete point when they assume the existence of a word f such that $a = f(b)$ because $a = \hat{\sigma}(-b)$. However, we will see there always exists a word f' such that $-b = f'(b)$ for any b.

(b) $x \in B_{2k}$ and $\sigma(x) = y \in B_{2k+1}$

Since $-x \in B_{2k-1} \cup B_{2k} \cup B_{2k+1}$ (lemma 3) and $\bar{\sigma}(-x) = y$, we can conclude that $-x$ cannot be an element of $B_{2k-1} \bigcup B_{2k}$ because $\bar{\sigma}$ saturates this set. Furthermore, we can use the idea of case (a) to prove that $-x, y \in B_{2k+1}$ are linked in $\{\hat{\sigma}, \sigma\}$. Since x links to $-x$, we obtain that x links to y.

(c) $x \in B_{2k+1}$ and $\sigma(x) = y \in B_{2k}$

Assuming that not all such x and y are linked, we can consider without loss of generality that $n = 2k + 1$ is the maximum element in $\{n' \in \mathbb{N} : \exists x_{n'} \in B_{n'}$ such that $\sigma(x_{n'}) \in B_{n'-1}$ is not linked to x'_n *or* $\bar{\sigma}(x_{n'}) \in B_{n'-1}$ is not linked to $x'_n\}$. If this maximum does not have the form $2k+1$, the case can be treated similarly by finding the same contradiction on $\bar{\sigma}$.

Lets denote by z the last element of B_{2k} in the sequence $y, \sigma(y), \sigma^2(y), \ldots, \sigma^i(y) = x$. Since $\hat{\sigma}(z) = \sigma_{/B_{2k}}(z) = y$ (because, from the choice of z, all $\sigma(z), \sigma^2(z), \ldots x$ are in B_{2k+1}), we can directly see that y and z are linked. Since z and $-z$ are linked, we obtain that y and $-z$ are linked and we are going to show that $-z$ and x are also linked. Since $\bar{\sigma}(-z) = \sigma(z) \in B_{2k+1}$, and because $\bar{\sigma}$ saturates $B_{2k+1} \cup B_{2k+2}$, there are two possibilities for the provenience of $-z$:

- $-z \in B_{2k+2} \Rightarrow -z$ and $\bar{\sigma}(-z)$ are linked because of the maximality property of $n = 2k + 1$
- $-z \in B_{2k+1} \Rightarrow \hat{\sigma}(-z) = \bar{\sigma}_{/B_{2k+1}}(-z) = \bar{\sigma}(-z)$ and thus $-z$ and $\bar{\sigma}(-z)$ are linked in $\{\hat{\sigma}, \sigma\}$

To finish, we still need to check that $\bar{\sigma}(-z) = \sigma(z)$ and x are linked. Since all elements of $\sigma(z), \sigma^2(z), \ldots \sigma^j(z) = x$ are in B_{2k+1}, it is enough to prove that $\forall t \in B_{2k+1}$ such that $\sigma(t) \in B_{2k+1}$, t and $\sigma(t)$ are linked. For this, it is enough to show that $-t$ and $\bar{\sigma}(-t) = \sigma(t)$ are linked. Since $\bar{\sigma}(-t) \in B_{2k+1}$, we can prove that $-t$ and $\bar{\sigma}(-t)$ are linked by the same procedure we proved that $-z$ and $\bar{\sigma}(-z)$ are linked.

(d) $x \in B_{2k+1}$ and $\sigma(x) = y \in B_{2k+1}$

This is the last possibility. Taking into account that $B_{2k+1} \bigcup B_{2k+2}$ is saturated by $\bar{\sigma}$ and that $\bar{\sigma}(-x) \in B_{2k+1}$, we can have either $-x \in B_{2k+2}$ or $-x \in B_{2k+1}$. One might now notice that the case $-x, b \in B_{2k+1}$ is the dual of (a) and the case $-x \in B_{2k+2}$, $\bar{\sigma}(-x) \in B_{2k+1}$ is a dual of (c).

2.1 Concluding Remarks

The next steps of the proof can be found in [5] and they follow quite naturally. To prove that the resulting map is a tree, one has to verify that $z(\bar{\hat{\sigma}}) = 1$, i.e. $\hat{\sigma}$ has only one cycle. To do this, one uses the fact that a planar map is defined as a map with genus 1, where the genus is a measure derived with from $z(\sigma)$ and $z(\bar{\sigma})$. The tree is well labeled if the difference between the labels on two connected vertices is 1. The well-labeling is based on the indexes of the B_i's (see Figure 2 in which the color of a vertex designates a number from the attached legend), i.e. one proves that any two neighboring vertices are in consecutive B_i's.

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