1 Equality

1.1 Syntax

1.1.1 Types

%datatype *type*
%name *type* τ

τ ::=
| **unit**
| $\tau_1 \to \tau_2$

1.1.2 Terms

%datatype *term*
%name *term* t

t ::=
| $\langle\rangle$
| $t_1\, t_2$
| $\lambda x{:}\,\tau.t$

%binding $1 \mapsto 3$ **in** $\lambda \sqcup : \sqcup . \sqcup$

1.2 Typing judgment

%judgment $\vdash t{:}\,\tau$

$$\frac{}{\vdash \langle\rangle{:}\,\mathbf{unit}}\,[\text{of\_empty}]$$

$$\frac{\{x\} \ \vdash x{:}\,\tau \ \rightarrow \ \vdash t[x]{:}\,\tau'}{\vdash \lambda x{:}\,\tau.t[x]{:}\,\tau \to \tau'}\,[\text{of\_lam}]$$

$$\frac{\vdash t_1{:}\,\tau \to \tau' \quad \vdash t_2{:}\,\tau}{\vdash t_1\, t_2{:}\,\tau'}\,[\text{of\_app}]$$

1.3 Values

%judgment t **value**

$$\langle\rangle \ \mathbf{value}\,[\text{value\_empty}]$$

$$\lambda x{:}\,\tau.t[x] \ \mathbf{value}\,[\text{value\_lam}]$$

1.4 Algorithmic definition of equality

%judgment $t_1 = t_2$

$$\frac{}{\langle\rangle = \langle\rangle}\;[\text{equ\_u}]$$

$$\frac{\{x\}\ x = x \ \rightarrow\ t[x] = t'[x]}{\lambda x{:}\,\tau.t[x] = \lambda x{:}\,\tau.t'[x]}\;[\text{equ\_l}]$$

$$\frac{t_1 = e_1 \quad t_2 = e_2}{(t_1\, t_2) = (e_1\, e_2)}\;[\text{equ\_a}]$$

1.5 Reflexivity is admissible (first attempt)

%lemma $\forall t{:}\, \mathit{term} \quad \vdash_{\text{ref}} \quad \exists \mathcal{D}{:}\, t = t$
%mode $+t \quad \vdash_{\text{ref}} \quad -\mathcal{D}$

$$\frac{}{\langle\rangle{:}\ \mathit{term} \quad \vdash_{\text{ref}} \quad \dfrac{}{\langle\rangle = \langle\rangle}\;[\text{equ\_u}]}\;[\text{r\_a}]$$

$$\frac{\{x\}\ \{\mathcal{U}{:}\, x = x\} \left(t[x]{:}\ \mathit{term} \quad \vdash_{\text{ref}} \quad \begin{matrix}\mathcal{D}\ x\ \mathcal{U}\\ t[x] = t[x]\end{matrix} \right)}{\begin{matrix}\mathcal{D}\\ \lambda x{:}\,\tau.t[x]{:}\ \mathit{term} \quad \vdash_{\text{ref}} \quad \dfrac{\{x\}\ x = x \ \rightarrow\ t[x] = t[x]}{\lambda x{:}\,\tau.t[x] = \lambda x{:}\,\tau.t[x]}\;[\text{equ\_l}]\end{matrix}}\;[\text{r\_l}]$$

$$\frac{\left(t_1{:}\ \mathit{term} \quad \vdash_{\text{ref}} \quad \begin{matrix}\mathcal{D}_1\\ t_1 = t_1\end{matrix} \right) \ \left(t_2{:}\ \mathit{term} \quad \vdash_{\text{ref}} \quad \begin{matrix}\mathcal{D}_2\\ t_2 = t_2\end{matrix} \right)}{(t_1\, t_2){:}\ \mathit{term} \quad \vdash_{\text{ref}} \quad \dfrac{\begin{matrix}\mathcal{D}_1\\ t_1 = t_1\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ t_2 = t_2\end{matrix}}{(t_1\, t_2) = (t_1\, t_2)}\;[\text{equ\_a}]}\;[\text{r\_a}]$$

%block $\mathcal{W}_{\text{ref}}$: **block** $\{x{:}\, \mathit{term}\}\{\mathcal{U}{:}\, x = x\}$
%worlds $(\mathcal{W}_{\text{ref}}) \quad t \quad \vdash_{\text{ref}} \quad \mathcal{D}$
%terminates $(t) \quad t \quad \vdash_{\text{ref}} \quad \mathcal{D}$

Remark. Checking this totality assertion yields "Coverage error":
%total $(t) \quad t \quad \vdash_{\text{ref}} \quad \mathcal{D}$

1.6 Reflexivity is admissible

%lemma $\forall t{:}\, \mathit{term} \quad \vdash_{\text{ref}} \quad \exists \mathcal{D}{:}\, t = t$
%mode $+t \quad \vdash_{\text{ref}} \quad -\mathcal{D}$

$$\frac{}{\langle\rangle{:}\ \mathit{term} \quad \vdash_{\text{ref}} \quad \dfrac{}{\langle\rangle = \langle\rangle}\;[\text{equ\_u}]}\;[\text{r\_u}]$$

$$\frac{\{x\}\ \{\mathcal{U}{:}\, x = x\} \left(x{:}\ \mathit{term} \quad \vdash_{\text{ref}} \quad \begin{matrix}\mathcal{U}\\ x = x\end{matrix} \right) \rightarrow \left(t[x]{:}\ \mathit{term} \quad \vdash_{\text{ref}} \quad \begin{matrix}\mathcal{D}\ x\ \mathcal{U}\\ t[x] = t[x]\end{matrix} \right)}{\begin{matrix}\mathcal{D}\\ \lambda x{:}\,\tau.t[x]{:}\ \mathit{term} \quad \vdash_{\text{ref}} \quad \dfrac{\{x\}\ x = x \ \rightarrow\ t[x] = t[x]}{\lambda x{:}\,\tau.t[x] = \lambda x{:}\,\tau.t[x]}\;[\text{equ\_l}]\end{matrix}}\;[\text{r\_l}]$$

$$\dfrac{\left(t_1\colon term \quad \vdash_{\mathrm{ref}} \quad \begin{matrix}\mathcal{D}_1\\ t_1 = t_1\end{matrix}\right) \quad \left(t_2\colon term \quad \vdash_{\mathrm{ref}} \quad \begin{matrix}\mathcal{D}_2\\ t_2 = t_2\end{matrix}\right)}{(t_1\, t_2)\colon term \quad \vdash_{\mathrm{ref}} \quad \dfrac{\begin{matrix}\mathcal{D}_1\\ t_1 = t_1\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ t_2 = t_2\end{matrix}}{(t_1\, t_2) = (t_1\, t_2)}\,[\mathrm{equ\_a}]}\,[\mathrm{r\_a}]$$

%block $\mathcal{W}_{\mathrm{ref}}$: **block** $\{x\colon term\}\{\mathcal{U}\colon x = x\}\{\sqcup : (x\colon term \quad \vdash_{\mathrm{ref}} \quad \mathcal{U}\colon x = x)\}$
%worlds $(\mathcal{W}_{\mathrm{ref}}) \quad t \quad \vdash_{\mathrm{ref}} \quad \mathcal{D}$
%terminates $(t) \quad t \quad \vdash_{\mathrm{ref}} \quad \mathcal{D}$
%total $(t) \quad t \quad \vdash_{\mathrm{ref}} \quad \mathcal{D}$

1.7 Transitivity is admissible

%lemma $t = r \quad \wedge \quad r = s \quad \vdash_{\mathrm{tr}} \quad t = s$
%mode $+\mathcal{E}_1 \quad \wedge \quad +\mathcal{E}_2 \quad \vdash_{\mathrm{tr}} \quad -\mathcal{E}_3$

$$\dfrac{}{\dfrac{}{\langle\rangle = \langle\rangle}\,[\mathrm{equ\_u}] \quad \wedge \quad \dfrac{}{\langle\rangle = \langle\rangle}\,[\mathrm{equ\_u}] \quad \vdash_{\mathrm{tr}} \quad \dfrac{}{\langle\rangle = \langle\rangle}\,[\mathrm{equ\_u}]}\,[\mathrm{tr\_u}]$$

$$\dfrac{\{x\}\ \{\mathcal{U}\colon x = x\}\ (\mathcal{U} \quad \wedge \quad \mathcal{U} \quad \vdash_{\mathrm{tr}} \quad \mathcal{U}) \rightarrow \left(\begin{matrix}\mathcal{E}_1\ x\ \mathcal{U}\\ t[x] = r[x]\end{matrix} \quad \wedge \quad \begin{matrix}\mathcal{E}_2\ x\ \mathcal{U}\\ r[x] = s[x]\end{matrix} \quad \vdash_{\mathrm{tr}} \quad \begin{matrix}\mathcal{E}_3\ x\ \mathcal{U}\\ t[x] = s[x]\end{matrix}\right)}{\dfrac{\mathcal{E}_1}{\lambda x\colon \tau.t[x] = \lambda x\colon \tau.r[x]}\,[\mathrm{equ\_l}] \quad \wedge \quad \dfrac{\mathcal{E}_2}{\lambda x\colon \tau.r[x] = \lambda x\colon \tau.s[x]}\,[\mathrm{equ\_l}] \quad \vdash_{\mathrm{tr}} \quad \dfrac{\mathcal{E}_3}{\lambda x\colon \tau.t[x] = \lambda x\colon \tau.s[x]}\,[\mathrm{equ\_l}]}\,[\mathrm{tr\_l}]$$

$$\dfrac{\left(\begin{matrix}\mathcal{E}_1\\ t_1 = r_1\end{matrix} \quad \wedge \quad \begin{matrix}\mathcal{F}_1\\ r_1 = s_1\end{matrix} \quad \vdash_{\mathrm{tr}} \quad \begin{matrix}\mathcal{G}_1\\ t_1 = s_1\end{matrix}\right) \quad \left(\begin{matrix}\mathcal{E}_2\\ t_2 = r_2\end{matrix} \quad \wedge \quad \begin{matrix}\mathcal{F}_2\\ r_2 = s_2\end{matrix} \quad \vdash_{\mathrm{tr}} \quad \begin{matrix}\mathcal{G}_2\\ t_2 = s_2\end{matrix}\right)}{\dfrac{\begin{matrix}\mathcal{E}_1\\ t_1 = r_1\end{matrix} \quad \begin{matrix}\mathcal{E}_2\\ t_2 = r_2\end{matrix}}{(t_1\, t_2) = (r_1\, r_2)}\,[\mathrm{equ\_a}] \quad \wedge \quad \dfrac{\begin{matrix}\mathcal{F}_1\\ r_1 = s_1\end{matrix} \quad \begin{matrix}\mathcal{F}_2\\ r_2 = s_2\end{matrix}}{(r_1\, r_2) = (s_1\, s_2)}\,[\mathrm{equ\_a}] \quad \vdash_{\mathrm{tr}} \quad \dfrac{\begin{matrix}\mathcal{G}_1\\ t_1 = s_1\end{matrix} \quad \begin{matrix}\mathcal{G}_2\\ t_2 = s_2\end{matrix}}{(t_1\, t_2) = (s_1\, s_2)}\,[\mathrm{equ\_a}]}\,[\mathrm{tr\_a}]$$

%block $\mathcal{W}_{\mathrm{tr}}$: **block** $\{x\colon term\}\{\mathcal{U}\colon x = x\}\{\sqcup : (\mathcal{U}\colon x = x \quad \wedge \quad \mathcal{U}\colon x = x \quad \vdash_{\mathrm{tr}} \quad \mathcal{U}\colon x = x)\}$
%worlds $(\mathcal{W}_{\mathrm{tr}}) \quad \mathcal{E}_1 \quad \wedge \quad \mathcal{E}_2 \quad \vdash_{\mathrm{tr}} \quad \mathcal{E}_3$
%terminates $(\mathcal{E}_1) \quad \mathcal{E}_1 \quad \wedge \quad \mathcal{E}_2 \quad \vdash_{\mathrm{tr}} \quad \mathcal{E}_3$
%total $(\mathcal{E}_1) \quad \mathcal{E}_1 \quad \wedge \quad \mathcal{E}_2 \quad \vdash_{\mathrm{tr}} \quad \mathcal{E}_3$

1.8 Symmetry is admissible

%lemma $t = r \quad \vdash_{\mathrm{sym}} \quad r = t$
%mode $+\mathcal{E}_1 \quad \vdash_{\mathrm{sym}} \quad -\mathcal{E}_2$

$$\dfrac{}{\dfrac{}{\langle\rangle = \langle\rangle}\,[\mathrm{equ\_u}] \quad \vdash_{\mathrm{sym}} \quad \dfrac{}{\langle\rangle = \langle\rangle}\,[\mathrm{equ\_u}]}\,[\mathrm{tr\_u}]$$

$$\dfrac{\{x\}\ \{\mathcal{U}\colon x = x\} \left(\begin{matrix}\mathcal{U}\\ x = x\end{matrix} \quad \vdash_{\mathrm{sym}} \quad \begin{matrix}\mathcal{U}\\ x = x\end{matrix}\right) \rightarrow \left(\begin{matrix}\mathcal{E}_1\ x\ \mathcal{U}\\ t[x] = r[x]\end{matrix} \quad \vdash_{\mathrm{sym}} \quad \begin{matrix}\mathcal{E}_1'\ x\ \mathcal{U}\\ r[x] = t[x]\end{matrix}\right)}{\dfrac{\begin{matrix}\mathcal{E}_1\\ \{x\}\ x = x \ \rightarrow\ t[x] = r[x]\end{matrix}}{\lambda x\colon \tau.t[x] = \lambda x\colon \tau.r[x]}\,[\mathrm{equ\_l}] \quad \vdash_{\mathrm{sym}} \quad \dfrac{\begin{matrix}\mathcal{E}_1'\\ \{x\}\ x = x \ \rightarrow\ r[x] = t[x]\end{matrix}}{\lambda x\colon \tau.r[x] = \lambda x\colon \tau.t[x]}\,[\mathrm{equ\_l}]}\,[\mathrm{tr\_l}]$$

$$\frac{\left(\begin{array}{ccc}\mathcal{E}_1 & & \mathcal{E}_1' \\ t_1 = r_1 & \vdash_{\text{sym}} & r_1 = t_1\end{array}\right) \quad \left(\begin{array}{ccc}\mathcal{E}_2 & & \mathcal{E}_2' \\ t_2 = r_2 & \vdash_{\text{sym}} & r_2 = t_2\end{array}\right)}{\dfrac{\begin{array}{cc}\mathcal{E}_1 & \mathcal{E}_2 \\ t_1 = r_1 & t_2 = r_2\end{array}}{(t_1\, t_2) = (r_1\, r_2)}\,[\text{equ\_a}] \quad \vdash_{\text{sym}} \quad \dfrac{\begin{array}{cc}\mathcal{E}_1' & \mathcal{E}_2' \\ r_1 = t_1 & r_2 = t_2\end{array}}{(r_1\, r_2) = (t_1\, t_2)}\,[\text{equ\_a}]}\,[\text{tr\_a}]$$

%block $\mathcal{W}_{\text{sym}}$: **block** $\{x\colon \mathit{term}\}\{\mathcal{U}\colon x = x\}\{\sqcup\colon (\mathcal{U}\colon x = x \quad \vdash_{\text{sym}} \quad \mathcal{U}\colon x = x)\}$
%worlds $(\mathcal{W}_{\text{sym}})$ $\mathcal{E}_1 \quad \vdash_{\text{sym}} \quad \mathcal{E}_2$
%terminates $(\mathcal{E}_1)$ $\mathcal{E}_1 \quad \vdash_{\text{sym}} \quad \mathcal{E}_2$
%total $(\mathcal{E}_1)$ $\mathcal{E}_1 \quad \vdash_{\text{sym}} \quad \mathcal{E}_2$