1 Computation and deduction (Chapter 6)

1.1 Syntax

Expressions

%datatype *exp*
%name *exp* E

$$E ::= \;|\; E_1\, E_2 \;|\; \mathbf{lam}\, x.E$$

%binding $1 \mapsto 2$ **in** $\mathbf{lam}\, \sqcup . \sqcup$

Modified de Bruijn Expressions

%datatype *exp'*
%name *exp'* F

$$F ::= \;|\; 1 \;|\; F\!\uparrow \;|\; \Lambda F \;|\; F_1\, F_2$$

Environments and values

%datatype *env*
%name *env* η

%datatype *val*
%name *val* W

$$\eta ::= \;|\; \cdot \;|\; \eta, W$$

$$W ::= \;|\; \{\eta; F\}$$

1.2 Judgments

%judgment $\vdash E_1 \hookrightarrow E_2$

$$\frac{}{\vdash \mathbf{lam}\, x.e[x] \hookrightarrow \mathbf{lam}\, x.e[x]}\;[\text{ev\_lam}]$$

$$\frac{\vdash e_1 \hookrightarrow \mathbf{lam}\, x.e_1'[x] \quad \vdash e_2 \hookrightarrow v_2 \quad \vdash e_1'[v_2] \hookrightarrow v}{\vdash e_1\, e_2 \hookrightarrow v}\;[\text{ev\_app}]$$

%mode $\vdash +E_1 \hookrightarrow -E_2$
%worlds $()$ $\vdash E_1 \hookrightarrow E_2$
%unique $\vdash +E_1 \hookrightarrow -1E_2$

%judgment $\eta \vdash F \hookrightarrow W$

$$\frac{}{\eta, W \vdash 1 \hookrightarrow W}\,[\text{fev\_1}]$$

$$\frac{\eta \vdash F \hookrightarrow W}{\eta, W \vdash F\!\uparrow \hookrightarrow W}\,[\text{fev\_}\!\uparrow]$$

$$\frac{}{\eta \vdash \Lambda F \hookrightarrow \{\eta; \Lambda F\}}\,[\text{fev\_lam}]$$

$$\frac{\eta \vdash F_1 \hookrightarrow \{\eta'; \Lambda F_1'\} \quad \eta \vdash F_2 \hookrightarrow W_2 \quad \eta', W_2 \vdash F_1' \hookrightarrow W}{\eta \vdash F_1\, F_2 \hookrightarrow W}\,[\text{fev\_app}]$$

%mode $+\eta \vdash +F \hookrightarrow -W$
%worlds $()$ $\eta \vdash F \hookrightarrow W$
%unique $+\eta \vdash +F \hookrightarrow -1W$

Example

%solve $\cdot \vdash (\Lambda(\Lambda(1\!\uparrow)))\,(\Lambda 1) \hookrightarrow W$

%judgment $\eta \vdash F \leftrightarrow E$
%judgment $W \Leftrightarrow E$

$$\frac{\mathbf{\{}w\mathbf{\}}\ \mathbf{\{}x\mathbf{\}}\ w \Leftrightarrow x \ \rightarrow\ \eta, w \vdash F \leftrightarrow e[x]}{\eta \vdash \Lambda F \leftrightarrow \mathbf{lam}\, x.e[x]}\,[\text{tr\_lam}]$$

$$\frac{\eta \vdash F_1 \leftrightarrow e_1 \quad \eta \vdash F_2 \leftrightarrow e_2}{\eta \vdash F_1\, F_2 \leftrightarrow e_1\, e_2}\,[\text{tr\_app}]$$

$$\frac{W \Leftrightarrow e}{\eta, W \vdash 1 \leftrightarrow e}\,[\text{tr\_1}]$$

$$\frac{\eta \vdash F \leftrightarrow e}{\eta, W \vdash F\!\uparrow \leftrightarrow e}\,[\text{tr\_}\!\uparrow]$$

$$\frac{\eta \vdash \Lambda F \leftrightarrow \mathbf{lam}\, x.e[x]}{\{\eta; \Lambda F\} \Leftrightarrow \mathbf{lam}\, x.e[x]}\,[\text{vtr\_lam}]$$

%block $\mathcal{W}_0$: **block** $\{w\colon \mathit{val}\}\{x\colon \mathit{exp}\}\{\sqcup : w \Leftrightarrow x\}$
%mode
$+\,\eta \vdash +F \leftrightarrow -e$
$+\,W \Leftrightarrow -v$
%worlds $(\mathcal{W}_0)$
$\eta \vdash F \leftrightarrow e$
$W \Leftrightarrow v$
%unique
$+\,\eta \vdash +F \leftrightarrow -1e$
$+\,W \Leftrightarrow -1v$

%solve $\cdot \vdash (\Lambda(\Lambda(1\!\uparrow)))\,(\Lambda 1) \leftrightarrow (\mathbf{lam}\, x.\mathbf{lam}\, y.x)\,(\mathbf{lam}\, v.v)$
%solve $\cdot \vdash F \leftrightarrow (\mathbf{lam}\, x.\mathbf{lam}\, y.x)\,(\mathbf{lam}\, v.v)$
%solve $\cdot \vdash (\Lambda(\Lambda(1\!\uparrow)))\,(\Lambda 1) \leftrightarrow e$

%judgment $\vdash e \hookrightarrow v \quad \wedge \quad \eta \vdash F \leftrightarrow e \quad \vdash_{\text{map}} \quad \eta \vdash F \hookrightarrow W \quad \wedge \quad W \Leftrightarrow v$

$$\frac{}{\cdot\,[\text{ev\_lam}] \quad \wedge \quad \frac{\frac{\frac{\mathcal{C}_2}{}\,[\text{tr\_lam}]}{}\,[\text{vtr\_lam}]}{}\,[\text{tr\_1}] \quad \vdash_{\text{map}} \quad \cdot\,[\text{fev\_1}] \quad \wedge \quad \frac{\frac{\mathcal{C}_2}{}\,[\text{tr\_lam}]}{}\,[\text{vtr\_lam}]}\,[\text{mp\_1}]$$

$$\frac{\mathcal{D} \quad \wedge \quad \mathcal{C}_1 \quad \vdash_{\text{map}} \quad \mathcal{D}_1' \quad \wedge \quad \mathcal{U}_1}{\mathcal{D} \quad \wedge \quad \dfrac{\mathcal{C}_1}{}\,\text{[tr\_}\uparrow\text{]} \quad \vdash_{\text{map}} \quad \dfrac{\mathcal{D}_1'}{}\,\text{[fev\_}\uparrow\text{]} \quad \wedge \quad \mathcal{U}_1}\,\text{[mp\_}\uparrow\text{]}$$

$$\frac{}{\cdot\ \text{[ev\_lam]} \quad \wedge \quad \dfrac{\mathcal{C}_1}{}\,\text{[tr\_lam]} \quad \vdash_{\text{map}} \quad \cdot\ \text{[fev\_lam]} \quad \wedge \quad \dfrac{\dfrac{\mathcal{C}_1}{}\,\text{[tr\_lam]}}{}\,\text{[vtr\_lam]}}\,\text{[mp\_lam]}$$

$$\frac{\begin{array}{c}\mathcal{D}_1 \quad \wedge \quad \mathcal{C}_1 \quad \vdash_{\text{map}} \quad \mathcal{D}_1' \quad \wedge \quad \dfrac{\dfrac{\mathcal{C}_3}{}\,\text{[tr\_lam]}}{}\,\text{[vtr\_lam]} \\ \mathcal{D}_2 \quad \wedge \quad \mathcal{C}_2 \quad \vdash_{\text{map}} \quad \mathcal{D}_2' \quad \wedge \quad \mathcal{U}_2 \\ \mathcal{D}_3 \quad \wedge \quad \mathcal{C}_3\, \mathcal{W}_2\, \mathcal{V}_2\, \mathcal{U}_2 \quad \vdash_{\text{map}} \quad \mathcal{D}_3' \quad \wedge \quad \mathcal{U}_3\end{array}}{\dfrac{\mathcal{D}_1 \quad \mathcal{D}_2 \quad \mathcal{D}_3}{}\,\text{[ev\_app]} \quad \wedge \quad \dfrac{\mathcal{C}_1 \quad \mathcal{C}_2}{}\,\text{[tr\_app]} \quad \vdash_{\text{map}} \quad \dfrac{\mathcal{D}_1' \quad \mathcal{D}_2' \quad \mathcal{D}_3'}{}\,\text{[fev\_app]} \quad \wedge \quad \mathcal{U}_3}\,\text{[mp\_app]}$$

Remark. The above rules correspond to the following Twelf code:

```
mp_1 : map_eval (ev_lam) (tr_1 (vtr_lam (tr_lam C2)))
               (fev_1) (vtr_lam (tr_lam C2)).

mp_uparrow : map_eval D (tr_uparrow C1) (fev_uparrow D1') U1
            <- map_eval D C1 D1' U1.

mp_lam : map_eval (ev_lam) (tr_lam C1)
               (fev_lam) (vtr_lam (tr_lam C1)).

mp_app : map_eval (ev_app D3 D2 D1) (tr_app C2 C1) (fev_app D3' D2' D1') U3
            <- map_eval D1 C1 D1' (vtr_lam (tr_lam C3))
            <- map_eval D2 C2 D2' U2
            <- map_eval D3 (C3 W2 V2 U2) D3' U3.
```

%mode $-\mathcal{D} \quad \wedge \quad +\mathcal{C} \quad \vdash_{\text{map}} \quad +\mathcal{D}' \quad \wedge \quad -\mathcal{U}$
%block $b_1 \quad : \quad$ **block** $\{x\colon exp\}$
%worlds $(b_1) \quad \mathcal{D} \quad \wedge \quad \mathcal{C} \quad \vdash_{\text{map}} \quad \mathcal{D}' \quad \wedge \quad \mathcal{U}$
%total $(\mathcal{D}') \quad \mathcal{D} \quad \wedge \quad \mathcal{C} \quad \vdash_{\text{map}} \quad \mathcal{D}' \quad \wedge \quad \mathcal{U}$

Remark. These properties do not hold:
%mode $+\mathcal{D} \quad \wedge \quad +\mathcal{C} \quad \vdash_{\text{map}} \quad -\mathcal{D}' \quad \wedge \quad -\mathcal{U}$
%terminates $(\mathcal{D}) \quad \mathcal{D} \quad \wedge \quad \mathcal{C} \quad \vdash_{\text{map}} \quad \mathcal{D}' \quad \wedge \quad \mathcal{U}$

%theorem thm :
$\forall^{\Gamma}($**pi** $\{x\colon exp\})$
$\forall^{*}\{e\}\{W\}\{F\}\{\eta\}$
$\forall\ \{\mathcal{D}'\colon \eta \vdash F \hookrightarrow W\}$
$\{\mathcal{C}\colon \eta \vdash F \leftrightarrow e\}$
$\exists\ \{v\}\{\mathcal{D}\colon \vdash e \hookrightarrow v\}\{\mathcal{U}\colon W \Leftrightarrow v\}$

Remark. The theorem prover loops:
%prove $5 \quad \mathcal{D}' \quad (\text{thm}\ \mathcal{D}'\ \mathcal{C}\ v\ \mathcal{D}\ \mathcal{U})$