1 Simply typed λ-calculus

1.1 Syntax

1.1.1 Types

%datatype *type*
%name *type* τ

$$
\begin{array}{lll}
\tau & ::= & \\
 & | & \mathbf{unit} \\
 & | & \tau_1 \to \tau_2
\end{array}
$$

1.1.2 Terms

%datatype *term*
%name *term* t

$$
\begin{array}{lll}
t & ::= & \\
 & | & \langle\rangle \\
 & | & t_1\, t_2 \\
 & | & \lambda x{:}\,\tau.t \\
 & | & [x \leftarrow t_1]t_2
\end{array}
$$

%binding $1 \mapsto 3$ **in** $\lambda_\sqcup :\, _\sqcup . _\sqcup$
%binding $1 \mapsto 3$ **in** $[_\sqcup \leftarrow _\sqcup]_\sqcup$

1.2 Typing judgment

%judgment $\vdash t{:}\,\tau$

$$\frac{}{\vdash \langle\rangle{:}\, \mathbf{unit}}\ [\text{of\_empty}]$$

$$\frac{\{x\}\ \vdash x{:}\,\tau\ \ \to\ \ \vdash t[x]{:}\,\tau'}{\vdash \lambda x{:}\,\tau.t[x]{:}\,\tau \to \tau'}\ [\text{of\_lam}]$$

$$\frac{\vdash t_1{:}\,\tau \to \tau' \quad \vdash t_2{:}\,\tau}{\vdash t_1\, t_2{:}\,\tau'}\ [\text{of\_app}]$$

1.3 Values

%judgment t **value**

$$\langle\rangle\ \ \mathbf{value}\ ^{[\text{value\_empty}]}$$

$$\lambda x{:}\,\tau.t[x]\ \ \mathbf{value}\ ^{[\text{value\_lam}]}$$

1.4 Equality

%judgment $t_1 = t_2$

$$t = t \;^{[\text{equal\_refl}]}$$

1.5 Inductively defined meta-substitution

%judgment $[x \leftarrow t_1]t_2 = t_3$
%binding $1 \mapsto 3$ **in** $[\sqcup \leftarrow \sqcup]\sqcup = \sqcup$

$$[x \leftarrow u]x = u \;^{[\text{subst\_var}]}$$

$$[x \leftarrow u]t = t \;^{[\text{subst\_term}]}$$

$$\frac{[x \leftarrow u](t_1[x]) = t_1' \quad [x \leftarrow u](t_2[x]) = t_2'}{[x \leftarrow u](t_1[x]\; t_2[x]) = t_1'\; t_2'}\;[\text{subst\_app}]$$

$$\frac{\mathbf{\{}y\mathbf{\}}\;\; [x \leftarrow u](t[y][x]) = t'[y]}{[x \leftarrow u](\lambda y{:}\,\tau.t[y][x]) = \lambda y{:}\,\tau.t'[y]}\;[\text{subst\_lam}]$$

%solve $[x \leftarrow \lambda y{:}\,\mathbf{unit}.y]x = \sqcup$
%solve $[x \leftarrow \lambda y{:}\,\mathbf{unit}.y]\lambda y{:}\,\mathbf{unit}.y\,x = \sqcup$

%block $\mathcal{W}$: **block** $\{x{:}\,\mathit{term}\}$
%mode $[\sqcup \leftarrow +t_1] + t_2 = -t_3$
%worlds $(\mathcal{W})$ $[\sqcup \leftarrow t_1]t_2 = t_3$
%terminates (t_2) $[\sqcup \leftarrow t_1]t_2 = t_3$

%lemma $(\mathbf{\{}y\mathbf{\}}\;\, t[y] = t'[y]) \quad \vdash_{\text{lam}} \quad \mathbf{\{}\tau\mathbf{\}}\;\, \lambda y{:}\,\tau.t[y] = \lambda y{:}\,\tau.t'[y]$

Proof.

$$\frac{}{\dfrac{\mathbf{[}y\mathbf{]}\;\dfrac{}{t[y] = t[y]}\;[\text{equal\_refl}]}{\mathbf{\{}y\mathbf{\}}\;\, t[y] = t[y]} \quad \vdash_{\text{lam}} \quad \dfrac{\mathbf{[}\tau\mathbf{]}\;\dfrac{}{\lambda y{:}\,\tau.t[y] = \lambda y{:}\,\tau.t[y]}\;[\text{equal\_refl}]}{\mathbf{\{}\tau\mathbf{\}}\;\, \lambda y{:}\,\tau.t[y] = \lambda y{:}\,\tau.t[y]}}\;[\&1]$$

%mode $+\mathcal{D}_1 \quad \vdash_{\text{lam}} \quad -\mathcal{D}_2$
%worlds $(\mathcal{W}) \quad \mathcal{D}_1 \quad \vdash_{\text{lam}} \quad \mathcal{D}_2$
%total $\{\} \quad \mathcal{D}_1 \quad \vdash_{\text{lam}} \quad \mathcal{D}_2$

%lemma $t_1 = t_1' \quad \wedge \quad t_2 = t_2' \quad \vdash_{\text{app}} \quad t_1\,t_2 = t_1'\,t_2'$

Proof.

$$\frac{}{t_1 = t_1 \;^{[\text{equal\_refl}]} \quad \wedge \quad t_2 = t_2 \;^{[\text{equal\_refl}]} \quad \vdash_{\text{app}} \quad t_1\,t_2 = t_1\,t_2 \;^{[\text{equal\_refl}]}}\;[\&1]$$

%mode $+\mathcal{D}_1 \quad \wedge \quad +\mathcal{D}_2 \quad \vdash_{\text{app}} \quad -\mathcal{D}_3$
%worlds $(\mathcal{W}) \quad \mathcal{D}_1 \quad \wedge \quad \mathcal{D}_2 \quad \vdash_{\text{app}} \quad \mathcal{D}_3$
%total $\{\} \quad \mathcal{D}_1 \quad \wedge \quad \mathcal{D}_2 \quad \vdash_{\text{app}} \quad \mathcal{D}_3$

%lemma $[x \leftarrow u](t_1[x]) = t_1' \quad \vdash_{\text{equiv}} \quad t_1[u] = t_1'$

Proof.

$$\frac{}{[x \leftarrow u]x = u \;^{[\text{subst\_var}]} \quad \vdash_{\text{equiv}} \quad u = u \;^{[\text{equal\_refl}]}}\;[\&1]$$

$$\frac{}{[x\leftarrow u]t=t^{\,\text{[subst\_term]}} \quad \vdash_{\text{equiv}} \quad t=t^{\,\text{[equal\_refl]}}}\;\text{[\&2]}$$

$$\frac{\begin{array}{c}
\begin{array}{c}\mathcal{D}_{s_1}\\ [x\leftarrow u](t_1[x])=t_1'\end{array} \quad \vdash_{\text{equiv}} \quad \begin{array}{c}\mathcal{D}_{e_1}\\ t_1[u]=t_1'\end{array}\\
\begin{array}{c}\mathcal{D}_{s_2}\\ [x\leftarrow u](t_2[x])=t_2'\end{array} \quad \vdash_{\text{equiv}} \quad \begin{array}{c}\mathcal{D}_{e_2}\\ t_2[u]=t_2'\end{array}\\
\begin{array}{c}\mathcal{D}_{e_1}\\ t_1[u]=t_1'\end{array} \quad \wedge \quad \begin{array}{c}\mathcal{D}_{e_2}\\ t_2[u]=t_2'\end{array} \quad \vdash_{\text{app}} \quad \begin{array}{c}\mathcal{D}_{e}\\ t_1[u]\,t_2[u]=t_1'\,t_2'\end{array}
\end{array}}{\dfrac{\begin{array}{c}\mathcal{D}_{s_1}\\ [x\leftarrow u](t_1[x])=t_1'\end{array} \quad \begin{array}{c}\mathcal{D}_{s_2}\\ [x\leftarrow u](t_2[x])=t_2'\end{array}}{[x\leftarrow u](t_1[x]\,t_2[x])=t_1'\,t_2'}\,\text{[subst_app]} \quad \vdash_{\text{equiv}} \quad \begin{array}{c}\mathcal{D}_{e}\\ t_1[u]\,t_2[u]=t_1'\,t_2'\end{array}}\;\text{[\&3]}$$

$$\frac{\begin{array}{c}
\{y\}\left(\begin{array}{c}\mathcal{D}_{s_1}\ y\\ [x\leftarrow u](t[y][x])=t'[y]\end{array} \quad \vdash_{\text{equiv}} \quad \begin{array}{c}\mathcal{D}_{e_1}\ y\\ t[y][u]=t'[y]\end{array}\right)\\
\begin{array}{c}\mathcal{D}_{e_1}\\ \{y\}\ t[y][u]=t'[y]\end{array} \quad \vdash_{\text{lam}} \quad \begin{array}{c}\mathcal{D}_{e_2}\\ \{\tau\}\ \lambda y\colon\tau.t[y][u]=\lambda y\colon\tau.t'[y]\end{array}
\end{array}}{\dfrac{\begin{array}{c}\mathcal{D}_{s_1}\\ \{y\}\ [x\leftarrow u](t[y][x])=t'[y]\end{array}}{[x\leftarrow u](\lambda y\colon\tau.t[y][x])=\lambda y\colon\tau.t'[y]}\,\text{[subst_lam]} \quad \vdash_{\text{equiv}} \quad \begin{array}{c}\mathcal{D}_{e_2}\ \tau\\ \lambda y\colon\tau.t[y][u]=\lambda y\colon\tau.t'[y]\end{array}}\;\text{[\&4]}$$

%mode $+\mathcal{D}_1 \quad \vdash_{\text{equiv}} \quad -\mathcal{D}_2$
%worlds $(\mathcal{W}) \quad \mathcal{D}_1 \quad \vdash_{\text{equiv}} \quad \mathcal{D}_2$
%terminates $(\mathcal{D}_1) \quad \mathcal{D}_1 \quad \vdash_{\text{equiv}} \quad \mathcal{D}_2$
%covers $+\mathcal{D}_1 \quad \vdash_{\text{equiv}} \quad -\mathcal{D}_2$
%total $(\mathcal{D}_1) \quad \mathcal{D}_1 \quad \vdash_{\text{equiv}} \quad \mathcal{D}_2$

1.6 Explicit substitution

1.6.1 One step reduction

%judgment $t_1 \rightsquigarrow t_2$

$$\frac{t_1 \rightsquigarrow t_1'}{t_1\,t_2 \rightsquigarrow t_1'\,t_2}\,\text{[step\_app}_1\text{]}$$

$$\frac{t_2 \rightsquigarrow t_2'}{t_1\,t_2 \rightsquigarrow t_1\,t_2'}\,\text{[step\_app}_2\text{]}$$

$$\frac{}{(\lambda x\colon\tau.t[x])\,u \rightsquigarrow [x\leftarrow u](t[x])}\,\text{[step\_beta]}$$

$$\frac{}{[x\leftarrow u]x \rightsquigarrow u}\,\text{[step\_subst\_var]}$$

$$\frac{}{[x\leftarrow u]t \rightsquigarrow t}\,\text{[step\_subst\_term]}$$

$$\frac{}{[x\leftarrow u](t_1[x]\,t_2[x]) \rightsquigarrow [x\leftarrow u](t_1[x])\,[x\leftarrow u](t_2[x])}\,\text{[step\_subst\_app]}$$

$$\frac{}{[x\leftarrow u](\lambda y\colon\tau.t[y][x]) \rightsquigarrow (\lambda y\colon\tau.[x\leftarrow u](t[y][x]))}\,\text{[step\_subst\_lam]}$$

1.6.2 Full evaluation

%judgment $t_1 \rightsquigarrow^\star t_2$

$$\frac{}{\lambda x\colon \tau . t[x] \rightsquigarrow^\star \lambda x\colon \tau . t[x]}\ [\text{step\_star}_1]$$

$$\frac{t \rightsquigarrow t' \quad t' \rightsquigarrow^\star t''}{t \rightsquigarrow^\star t''}\ [\text{step\_star}_2]$$

%solve $\lambda y\colon \mathbf{unit}. y\, \lambda y\colon \mathbf{unit}. y \rightsquigarrow^\star \sqcup$