1 Simply typed λ-calculus

1.1 Syntax

1.1.1 Types

$\mathit{type} \;:\; \mathbf{type}.$
$\mathbf{unit} \;:\; \mathit{type}.$
$\sqcup \to \sqcup \;:\; \mathit{type} \to \mathit{type} \to \mathit{type}.$
$\sqcup \times \sqcup \;:\; \mathit{type} \to \mathit{type} \to \mathit{type}.$

1.1.2 Terms

$\mathit{term} \;:\; \mathbf{type}.$

$\langle\rangle \;:\; \mathit{term}.$
$\sqcup\ \sqcup \;:\; \mathit{term} \to \mathit{term} \to \mathit{term}.$

$\lambda \sqcup : \sqcup . \sqcup \;:\; \mathit{type} \to (\mathit{term} \to \mathit{term}) \to \mathit{term}.$
$\mathbf{\%binding}\;\; 1 \mapsto 3 \;\;\mathbf{in}\;\; \lambda \sqcup : \sqcup . \sqcup$

$\langle \sqcup , \sqcup \rangle \;:\; \mathit{term} \to \mathit{term} \to \mathit{term}.$
$\mathbf{let}\ \langle \sqcup , \sqcup \rangle = \sqcup\ \mathbf{in}\ \sqcup \;:\; \mathit{term} \to (\mathit{term} \to \mathit{term} \to \mathit{term}) \to \mathit{term}.$
$\mathbf{\%binding}\;\; 2 \mapsto 4 \;\;\mathbf{in}\;\; \mathbf{let}\ \langle \sqcup , \sqcup \rangle = \sqcup\ \mathbf{in}\ \sqcup$
$\mathbf{\%binding}\;\; 1 \mapsto 4 \;\;\mathbf{in}\;\; \mathbf{let}\ \langle \sqcup , \sqcup \rangle = \sqcup\ \mathbf{in}\ \sqcup$

1.2 Typing judgment

$\vdash \sqcup : \sqcup \;:\; \mathit{term} \to \mathit{type} \to \mathbf{type}.$

$$\frac{}{\vdash \langle\rangle : \mathbf{unit}}\ [\text{of\_empty}]$$

$$\frac{\{x\}\ \vdash x : \tau \;\to\; \vdash t[x] : \tau'}{\vdash \lambda x : \tau . t[x] : \tau \to \tau'}\ [\text{of\_lam}]$$

$$\frac{\vdash t_1 \colon \tau \to \tau' \quad \vdash t_2 \colon \tau}{\vdash t_1\, t_2 \colon \tau'}\,[\text{of\_app}]$$

$$\frac{\vdash t_1 \colon \tau \quad \vdash t_2 \colon \tau'}{\vdash \langle t_1, t_2 \rangle \colon \tau \times \tau'}\,[\text{of\_pair}]$$

$$\frac{\{x\} \vdash x \colon \tau_1 \;\to\; (\{y\} \vdash y \colon \tau_2 \;\to\; \vdash u[x][y] \colon \tau) \quad \vdash t \colon \tau_1 \times \tau_2}{\vdash \mathbf{let}\ \langle x, y \rangle = t\ \mathbf{in}\ u[x][y] \colon \tau}\,[\text{of\_let}]$$

1.2.1 Derived meta-rule for the let macro

$(\{x\} \vdash x \colon \tau \;\to\; \vdash t[x] \colon \tau') \quad \bigwedge \quad \vdash t_2 \colon \tau \quad \Longrightarrow \quad \vdash \lambda x \colon \tau . t[x]\, t_2 \colon \tau' \quad : \quad \mathbf{type}.$

$$\frac{}{\overset{\mathcal{D}_{\text{of}_1}}{\{x\} \vdash x \colon \tau \;\to\; \vdash t[x] \colon \tau'} \quad \bigwedge \quad \overset{\mathcal{D}_{\text{of}_2}}{\vdash t_2 \colon \tau} \quad \Longrightarrow \quad \dfrac{\dfrac{\overset{\mathcal{D}_{\text{of}_1}}{\{x\} \vdash x \colon \tau \;\to\; \vdash t[x] \colon \tau'}}{\vdash \lambda x \colon \tau . t[x] \colon \tau \to \tau'}\,[\text{of\_lam}] \quad \overset{\mathcal{D}_{\text{of}_2}}{\vdash t_2 \colon \tau}}{\vdash \lambda x \colon \tau . t[x]\, t_2 \colon \tau'}\,[\text{of\_app}]}\,[\&]$$

%mode $+\mathcal{D}_{\text{of}_1} \quad \bigwedge \quad +\mathcal{D}_{\text{of}_2} \quad \Longrightarrow \quad -\mathcal{D}_{\text{of}_3}$
%worlds $() \quad \mathcal{D}_{\text{of}_1} \quad \bigwedge \quad \mathcal{D}_{\text{of}_2} \quad \Longrightarrow \quad \mathcal{D}_{\text{of}_3}$
%total $\{\} \quad \mathcal{D}_{\text{of}_1} \quad \bigwedge \quad \mathcal{D}_{\text{of}_2} \quad \Longrightarrow \quad \mathcal{D}_{\text{of}_3}$

1.3 Values

␣ **value** : *term* $\to$ **type**.

$$\langle\rangle\ \mathbf{value}^{\,[\text{value\_empty}]}$$

$$\lambda x \colon \tau . t[x]\ \mathbf{value}^{\,[\text{value\_lam}]}$$

$$\frac{t_1\ \mathbf{value} \quad t_2\ \mathbf{value}}{\langle t_1, t_2\rangle\ \mathbf{value}}\ [\text{value\_pair}]$$

1.4 Reduction semantics

$\_\!\rightarrow\!\_ \quad : \quad \mathit{term} \ \longrightarrow \ \mathit{term} \ \longrightarrow \ \mathbf{type}.$

$$\frac{t_1 \rightarrow t_1'}{t_1\, t_2 \rightarrow t_1'\, t_2}\ [\text{step\_app1}]$$

$$\frac{t_1\ \mathbf{value} \quad t_2 \rightarrow t_2'}{t_1\, t_2 \rightarrow t_1\, t_2'}\ [\text{step\_app2}]$$

$$\frac{t_1 \rightarrow t_1'}{\langle t_1, t_2\rangle \rightarrow \langle t_1', t_2\rangle}\ [\text{step\_pair1}]$$

$$\frac{t_1\ \mathbf{value} \quad t_2 \rightarrow t_2'}{\langle t_1, t_2\rangle \rightarrow \langle t_1, t_2'\rangle}\ [\text{step\_pair2}]$$

$$\frac{t_2\ \mathbf{value}}{\lambda x{:}\,\tau.t[x]\, t_2 \rightarrow t[t_2]}\ [\text{step\_beta}]$$

$$\frac{t_1\ \mathbf{value} \quad t_2\ \mathbf{value}}{\mathbf{let}\ \langle x, y\rangle = \langle t_1, t_2\rangle\ \mathbf{in}\ u[x][y] \rightarrow u[t_1][t_2]}\ [\text{step\_let}]$$

1.5 Preservation theorem

$t \rightarrow t' \quad \bigwedge \quad \vdash t{:}\,\tau \quad \Longrightarrow \quad \vdash t'{:}\,\tau \quad : \quad \mathbf{type}.$

$$\dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{stepE}_1}\\ t_1 \to t_1'\end{array} \;\bigwedge\; \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_1}\\ \vdash t_1 \colon \tau_2 \to \tau\end{array} \;\Longrightarrow\; \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_1'}\\ \vdash t_1' \colon \tau_2 \to \tau\end{array}}{\dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{stepE}_1}\\ t_1 \to t_1'\end{array}}{t_1\,t_2 \to t_1'\,t_2}[\text{step\_app1}] \;\bigwedge\; \dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{ofE}_1}\\ \vdash t_1 \colon \tau_2 \to \tau\end{array} \quad \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_2}\\ \vdash t_2 \colon \tau_2\end{array}}{\vdash t_1\,t_2 \colon \tau}[\text{of\_app}] \;\Longrightarrow\; \dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{ofE}_1'}\\ \vdash t_1' \colon \tau_2 \to \tau\end{array} \quad \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_2}\\ \vdash t_2 \colon \tau_2\end{array}}{\vdash t_1'\,t_2 \colon \tau}[\text{of\_app}]}[\&_1]$$

$$\dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{stepE}_2}\\ t_2 \to t_2'\end{array} \;\bigwedge\; \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_2}\\ \vdash t_2 \colon \tau_2\end{array} \;\Longrightarrow\; \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_2'}\\ \vdash t_2' \colon \tau_2\end{array}}{\dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{valE}_1}\\ t_1 \;\textbf{value}\end{array} \quad \begin{array}{c}\mathcal{D}_{\mathrm{stepE}_2}\\ t_2 \to t_2'\end{array}}{t_1\,t_2 \to t_1\,t_2'}[\text{step\_app2}] \;\bigwedge\; \dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{ofE}_1}\\ \vdash t_1 \colon \tau_2 \to \tau\end{array} \quad \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_2}\\ \vdash t_2 \colon \tau_2\end{array}}{\vdash t_1\,t_2 \colon \tau}[\text{of\_app}] \;\Longrightarrow\; \dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{ofE}_1}\\ \vdash t_1 \colon \tau_2 \to \tau\end{array} \quad \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_2'}\\ \vdash t_2' \colon \tau_2\end{array}}{\vdash t_1\,t_2' \colon \tau}[\text{of\_app}]}[\&_2]$$

$$\dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{stepE}_1}\\ t_1 \to t_1'\end{array} \;\bigwedge\; \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_1}\\ \vdash t_1 \colon \tau_1\end{array} \;\Longrightarrow\; \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_1'}\\ \vdash t_1' \colon \tau_1\end{array}}{\dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{stepE}_1}\\ t_1 \to t_1'\end{array}}{\langle t_1, t_2\rangle \to \langle t_1', t_2\rangle}[\text{step\_pair1}] \;\bigwedge\; \dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{ofE}_1}\\ \vdash t_1 \colon \tau_1\end{array} \quad \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_2}\\ \vdash t_2 \colon \tau_2\end{array}}{\vdash \langle t_1, t_2\rangle \colon \tau_1 \times \tau_2}[\text{of\_pair}] \;\Longrightarrow\; \dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{ofE}_1'}\\ \vdash t_1' \colon \tau_1\end{array} \quad \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_2}\\ \vdash t_2 \colon \tau_2\end{array}}{\vdash \langle t_1', t_2\rangle \colon \tau_1 \times \tau_2}[\text{of\_pair}]}[\&_3]$$

$$\dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{stepE}_2}\\ t_2 \to t_2'\end{array} \;\bigwedge\; \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_2}\\ \vdash t_2 \colon \tau_2\end{array} \;\Longrightarrow\; \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_2'}\\ \vdash t_2' \colon \tau_2\end{array}}{\dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{valE}_1}\\ t_1 \;\textbf{value}\end{array} \quad \begin{array}{c}\mathcal{D}_{\mathrm{stepE}_2}\\ t_2 \to t_2'\end{array}}{\langle t_1, t_2\rangle \to \langle t_1, t_2'\rangle}[\text{step\_pair2}] \;\bigwedge\; \dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{ofE}_1}\\ \vdash t_1 \colon \tau_1\end{array} \quad \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_2}\\ \vdash t_2 \colon \tau_2\end{array}}{\vdash \langle t_1, t_2\rangle \colon \tau_1 \times \tau_2}[\text{of\_pair}] \;\Longrightarrow\; \dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{ofE}_1}\\ \vdash t_1 \colon \tau_1\end{array} \quad \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_2'}\\ \vdash t_2' \colon \tau_2\end{array}}{\vdash \langle t_1, t_2'\rangle \colon \tau_1 \times \tau_2}[\text{of\_pair}]}[\&_4]$$

$$\dfrac{}{\dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{val}_2}\\ t_2 \;\textbf{value}\end{array}}{\lambda x \colon \tau_2 . t[x]\; t_2 \to t[t_2]}[\text{step\_beta}] \;\bigwedge\; \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{ofE}}\\ \boldsymbol{\{x\}}\; \vdash x \colon \tau_2 \;\boldsymbol{\to}\; \vdash t[x] \colon \tau\end{array}}{\vdash \lambda x \colon \tau_2 . t[x] \colon \tau_2 \to \tau}[\text{of\_lam}] \quad \begin{array}{c}\mathcal{D}_{\mathrm{ofE}_2}\\ \vdash t_2 \colon \tau_2\end{array}}{\vdash (\lambda x \colon \tau_2 . t[x])\; t_2 \colon \tau}[\text{of\_app}] \;\Longrightarrow\; \begin{array}{c}\mathcal{D}_{\mathrm{ofE}}\; t_2\; \mathcal{D}_{\mathrm{ofE}_2}\\ \vdash t[t_2] \colon \tau\end{array}}[\&_5]$$

$$\overline{\hspace{40em}}\;[\&_6]$$

$$\dfrac{\begin{array}{cc}\mathcal{D}_{\mathrm{val}_1} & \mathcal{D}_{\mathrm{val}_2}\\ t_1\ \textbf{value} & t_2\ \textbf{value}\end{array}}{\textbf{let}\ \langle x, y\rangle = \langle t_1, t_2\rangle\ \textbf{in}\ u[x][y] \to u[t_1][t_2]}\,[\text{step\_let}] \quad \bigwedge \quad \dfrac{\begin{array}{c}\mathcal{D}_{\mathrm{of}}\\ \{x\}\ \vdash x{:}\,\tau_1 \ \to\ (\{y\}\ \vdash y{:}\,\tau_2 \ \to\ \vdash u[x][y]{:}\,\tau)\end{array} \quad \dfrac{\begin{array}{cc}\mathcal{D}_{\mathrm{of}_1} & \mathcal{D}_{\mathrm{of}_2}\\ \vdash t_1{:}\,\tau_1 & \vdash t_2{:}\,\tau_2\end{array}}{\vdash \langle t_1, t_2\rangle{:}\,\tau_1 \times \tau_2}\,[\text{of\_pair}]}{\vdash \textbf{let}\ \langle x, y\rangle = \langle t_1, t_2\rangle\ \textbf{in}\ u[x][y]{:}\,\tau}\,[\text{of\_let}] \quad \Longrightarrow \quad \begin{array}{c}\mathcal{D}_{\mathrm{of}}\ t_1\ \mathcal{D}_{\mathrm{of}_1}\ t_2\ \mathcal{D}_{\mathrm{of}_2}\\ \vdash u[t_1][t_2]{:}\,\tau\end{array}$$

$$\begin{array}{l}
\textbf{\%mode}\quad +\mathcal{D}_{\mathrm{step}}\quad \bigwedge \quad +\mathcal{D}_{\mathrm{of}} \quad\Longrightarrow\quad -\mathcal{D}_{\mathrm{of}'}\\
\textbf{\%worlds}\quad ()\quad \mathcal{D}_{\mathrm{step}}\quad \bigwedge\quad \mathcal{D}_{\mathrm{of}}\quad \Longrightarrow\quad \mathcal{D}_{\mathrm{of}'}\\
\textbf{\%total}\quad \mathcal{D}_{\mathrm{step}}\quad \mathcal{D}_{\mathrm{step}}\quad \bigwedge\quad \mathcal{D}_{\mathrm{of}}\quad \Longrightarrow\quad \mathcal{D}_{\mathrm{of}'}
\end{array}$$