1 System T

1.1 Syntax

$idents \quad : \quad \textbf{type}.$
$term \quad : \quad \textbf{type}.$
$terms \quad : \quad \textbf{type}.$
$type \quad : \quad \textbf{type}.$
$types \quad : \quad \textbf{type}.$
$fenv \quad : \quad \textbf{type}.$

1.1.1 Identifiers

$\quad : \quad idents.$
$\textvisiblespace\,,\,\textvisiblespace \quad : \quad idents \ \rightarrow \ ident \ \rightarrow \ idents.$

1.1.2 Environment Σ

$\{\} \quad : \quad fenv.$
$\textvisiblespace\,,\textvisiblespace : \textvisiblespace \quad : \quad fenv \ \rightarrow \ ident \ \rightarrow \ type \ \rightarrow \ fenv.$

1.1.3 Term t

$x \quad : \quad term.$
$0 \quad : \quad term.$
$t_1\, t_2 \quad : \quad term.$
$\mathbf{fn}\, x{:}\, \tau \Rightarrow t \quad : \quad term.$
$\mathbf{succ}(\textvisiblespace) \quad : \quad term \ \rightarrow \ term.$
$\mathbf{pred}(\textvisiblespace) \quad : \quad term \ \rightarrow \ term.$
$\mathbf{rec}(t_1, t_2, t_3) \quad : \quad term.$
$\mathbf{let}\ x = t_1\ \mathbf{in}\ t_2 \quad : \quad term.$
$\langle \textvisiblespace \rangle \quad : \quad terms \ \rightarrow \ term.$
$\mathbf{let}\ \langle \textvisiblespace \rangle = \textvisiblespace\ \mathbf{in}\ \textvisiblespace \quad : \quad idents \ \rightarrow \ term \ \rightarrow \ term \ \rightarrow \ term.$

$\mathbf{fn}\,(\vec{x} : \vec{\tau}\,) \Rightarrow t \quad = \quad \mathbf{fn}\, \mathsf{z}{:}\, \vec{\tau} \Rightarrow (\mathbf{let}\ \langle \vec{x}\,\rangle = \mathsf{z}\ \mathbf{in}\ t).$

1.1.4 Terms

$\quad : \quad terms.$
$\textvisiblespace\,,\,\textvisiblespace \quad : \quad terms \ \rightarrow \ term \ \rightarrow \ terms.$

1.1.5 Type τ

$\top \quad : \quad type.$
$\bot \quad : \quad type.$
$\mathbf{nat} \quad : \quad type.$
$\textvisiblespace \rightarrow \textvisiblespace \quad : \quad type \ \rightarrow \ type \ \rightarrow \ type.$
$\sim \textvisiblespace \quad : \quad type \ \rightarrow \ type.$

$\langle \sqcup \rangle$: *types* $\rightarrow$ *type*.

1.1.6 Types $\vec{\tau}$

$\emptyset$: *types*.
$\vec{\tau}, \tau$: *types*.

1.2 Typing

Formulas equality

$\sqcup = \sqcup$: *type* $\rightarrow$ *type* $\rightarrow$ **type**.

$$\frac{}{\tau = \tau}\ [\text{form\_eq\_refl}]$$

Terms equality

$t_1 = t_2$: **type**.

$$\frac{}{t = t}\ [\text{term\_eq\_refl}]$$

Lookup

$\sqcup : \sqcup \in \sqcup$: *ident* $\rightarrow$ *type* $\rightarrow$ *fenv* $\rightarrow$ **type**.

$$\frac{}{x : \tau \in \Sigma, x : \tau}\ [\text{f\_lookup\_i}]$$

$$\frac{x \neq y \quad x : \tau \in \Sigma}{x : \tau \in \Sigma, y : \tau'}\ [\text{f\_lookup\_ii}]$$

Typing judgments

$\Sigma \vdash t : \tau$: **type**.
$\Sigma \vdash (\vec{t}\,) : (\vec{\tau}\,)$: **type**.
$\Sigma_1, \vec{x} : \vec{\tau} = \Sigma_2$: **type**.
$\Sigma, \langle \vec{x}\,\rangle : \langle \vec{\tau}\,\rangle \vdash t : \tau$: **type**.

Type check

$$\frac{x : \tau \in \Sigma}{\Sigma \vdash x : \tau}\ [\text{tc\_var}]$$

$$\frac{}{\Sigma \vdash 0 : \mathbf{nat}}\ [\text{tc\_zero}]$$

$$\frac{\Sigma \vdash t : \mathbf{nat}}{\Sigma \vdash \mathbf{succ}(t) : \mathbf{nat}}\ [\text{tc\_succ}]$$

$$\frac{\Sigma \vdash t : \mathbf{nat}}{\Sigma \vdash \mathbf{pred}(t) : \mathbf{nat}}\ [\text{tc\_pred}]$$

$$\frac{\Sigma, x\colon\tau \vdash t\colon\tau'}{\Sigma \vdash \mathbf{fn}\, x\colon\tau \Rightarrow t\colon \tau \to \tau'}\ \text{[tc\_lam]}$$

$$\frac{\Sigma \vdash t_1\colon \tau \to \tau' \quad \Sigma \vdash t_2\colon \tau}{\Sigma \vdash t_1\, t_2\colon \tau'}\ \text{[tc\_app]}$$

$$\frac{\Sigma \vdash t_1\colon \mathbf{nat} \quad \Sigma \vdash t_2\colon \tau \quad \Sigma \vdash t_3\colon \mathbf{nat} \to (\tau \to \tau)}{\Sigma \vdash \mathbf{rec}(t_1, t_2, t_3)\colon \tau}\ \text{[tc\_rec]}$$

$$\frac{\Sigma \vdash (\vec{t}\,)\colon (\vec{\tau}\,)}{\Sigma \vdash \langle \vec{t}\,\rangle\colon \langle \vec{\tau}\,\rangle}\ \text{[tc\_tuple]}$$

$$\frac{\Sigma \vdash t_1\colon \tau \quad \Sigma, y\colon\tau \vdash t_2\colon \tau'}{\Sigma \vdash \mathbf{let}\ y = t_1\ \mathbf{in}\ t_2\colon \tau'}\ \text{[tc\_let]}$$

$$\frac{\Sigma \vdash t_1\colon \langle \vec{\tau}\,\rangle \quad \Sigma, \langle \vec{x}\,\rangle\colon \langle \vec{\tau}\,\rangle \vdash t_2\colon \tau'}{\Sigma \vdash \mathbf{let}\ \langle \vec{x}\,\rangle = t_1\ \mathbf{in}\ t_2\colon \tau'}\ \text{[tc\_match]}$$

Append

$$\frac{}{\Sigma, ()\colon () = \Sigma}\ \text{[app\_i]}$$

$$\frac{\Sigma, \vec{x}\colon \vec{\tau} = \Sigma'}{\Sigma, (\vec{x}\,, x)\colon (\vec{\tau}\,, \tau) = \Sigma', x\colon \tau}\ \text{[app\_ii]}$$

Type check terms in extended environment

$$\frac{\Sigma, \vec{x}\colon \vec{\tau} = \Sigma' \quad \Sigma' \vdash t\colon \tau'}{\Sigma, \langle \vec{x}\,\rangle\colon \langle \vec{\tau}\,\rangle \vdash t\colon \tau'}\ \text{[tcte\_product]}$$

Type check terms

$$\frac{}{\Sigma \vdash ()\colon ()}\ \text{[tcts\_empty]}$$

$$\frac{\Sigma \vdash t\colon \tau \quad \Sigma \vdash (\vec{t}\,)\colon (\vec{\tau}\,)}{\Sigma \vdash (\vec{t}\,, t)\colon (\vec{\tau}\,, \tau)}\ \text{[tcts\_cons]}$$

1.3 Properties

%mode $+\tau_1 = +\tau_2$
%mode $+t_1 = +t_2$
%mode $+x\colon -\tau \in +\Sigma$
%mode $+\Sigma_1, +\vec{x}\colon +\vec{\tau} = -\Sigma_2$
%mode
$+\Sigma \vdash +t\colon -\tau$
$+\Sigma, \langle +\vec{x}\,\rangle\colon \langle +\vec{\tau}\,\rangle \vdash +t\colon -\tau'$
$+\Sigma \vdash (+\vec{t}\,)\colon (-\vec{\tau}\,)$

1.4 Examples

$t_0 \quad = \quad 0.$
%solve $\{\} \vdash t_0\colon \mathbf{nat}$
%solve $\{\} \vdash \mathbf{fn}\, \mathsf{x}\colon \mathbf{nat} \Rightarrow \mathbf{succ}(0)\colon \tau$
%solve $\{\} \vdash \mathbf{fn}\, \mathsf{x}\colon \mathbf{nat} \Rightarrow \mathbf{succ}(0)\colon \mathbf{nat} \to \mathbf{nat}$
%solve $\{\} \vdash \mathbf{fn}\, \mathsf{x}\colon \mathbf{nat} \Rightarrow \mathbf{fn}\, \mathsf{y}\colon \mathbf{nat} \Rightarrow \mathbf{rec}(\mathsf{x}, \mathsf{y}, \mathbf{fn}\, \mathsf{k}\colon \mathbf{nat} \Rightarrow \mathbf{fn}\, \mathsf{z}\colon \mathbf{nat} \Rightarrow \mathbf{succ}(0))\colon \tau$
%solve $\{\} \vdash \mathbf{fn}\, \mathsf{x}\colon \mathbf{nat} \Rightarrow \mathsf{x}\colon \mathbf{nat} \to \mathbf{nat}$