1 First simulation (soundness)

1.1 Syntax

1.1.1 Index, indices and tables

%datatype *index*
%name *index* n α

$$\begin{array}{lll} n & ::= & 0 \\ & | & n+1 \end{array}$$

%datatype *vector*
%name *vector* $\mathcal{I}$

$$\begin{array}{lll} \mathcal{I} & ::= & [\,] \\ & | & n :: \mathcal{I} \end{array}$$

%datatype *table*
%name *table* $\mathcal{I}_\mu$

$$\begin{array}{lll} \mathcal{I}_\mu & ::= & [\,] \\ & | & \mathcal{I} :: \mathcal{I}_\mu \end{array}$$

1.1.2 Term

%datatype *term*
%name *term* t

$$\begin{array}{lll} t & ::= & n \\ & | & t_1\, t_2 \\ & | & \lambda t \\ & | & \mathbf{catch}\, t \\ & | & \mathbf{throw}\, \alpha\, t \end{array}$$

Remark. Syntax of safe λ_{ct}-terms:

$$\begin{array}{lll} t & ::= & n \\ & | & t_1\, t_2 \\ & | & \lambda t \\ & | & \mathbf{get\text{-}context}\, t \\ & | & \mathbf{set\text{-}context}\, \alpha\, t \end{array}$$

1.2 Subtraction

%judgment $n_1 \dot- n_2 = n_3$

$$n_1 \dot- 0 = n_1 \;{}^{[\text{minus}_1]}$$
$$(n_1+1) \dot- (n_2+1) = n_3 \;{}^{[\text{minus}_2]} \quad \textit{when} \quad n_1 \dot- n_2 = n_3$$

%mode $+n_1 \dot- +n_2 = -n_3$
%worlds $()$ $n_1 \dot- n_2 = n_3$
%terminates (n_1) $n_1 \dot- n_2 = n_3$
%unique $+n_1 \dot- +n_2 = -1n_3$

%lemma $\forall n\colon \textit{index}\ \cdot\ n \dot- n = 0 \quad {}^{[\text{minus}\cdot\text{id}]}$

Proof.

$$\frac{}{0\colon \textit{index}\ \cdot\ 0 \dot- 0 = 0\;{}^{[\text{minus}_1]} \quad {}^{[\text{minus}\cdot\text{id}]}}\,[\&1]$$

$$\frac{n\colon \textit{index}\ \cdot\ \begin{array}{c}\mathcal{D}\\ n \dot- n = 0\end{array} \quad {}^{[\text{minus}\cdot\text{id}]}}{n+1\colon \textit{index}\ \cdot\ \begin{array}{c}\mathcal{D}\\ \dfrac{n \dot- n = 0}{(n+1) \dot- (n+1) = 0}\,[\text{minus}_2]\end{array} \quad {}^{[\text{minus}\cdot\text{id}]}}\,[\&2]$$

%mode $+n\ \cdot\ -\mathcal{D} \quad {}^{[\text{minus}\cdot\text{id}]}$
%worlds $()$ $n\ \cdot\ \mathcal{D} \quad {}^{[\text{minus}\cdot\text{id}]}$
%total (n) $n\ \cdot\ \mathcal{D} \quad {}^{[\text{minus}\cdot\text{id}]}$ $\square$

%lemma $n_1 \dot- (n_2+1) = n_3 \quad \Rightarrow \quad n_1 \dot- n_2 = (n_3+1) \quad {}^{[\text{minus}\cdot\text{succ}]}$

Proof.

$$\frac{}{\begin{array}{c}\mathcal{D}\\ (n_1+1) \dot- (0+1) = n_1\end{array} \quad \Rightarrow \quad (n_1+1) \dot- 0 = (n_1+1)\;{}^{[\text{minus}_1]} \quad {}^{[\text{minus}\cdot\text{succ}]}}\,[\&1]$$

$$\frac{\begin{array}{c}\mathcal{D}_1\\ n_1 \dot- (n_2+1) = n_3\end{array} \quad \Rightarrow \quad \begin{array}{c}\mathcal{D}_2\\ n_1 \dot- n_2 = (n_3+1)\end{array} \quad {}^{[\text{minus}\cdot\text{succ}]}}{\begin{array}{c}\mathcal{D}_1\\ \dfrac{n_1 \dot- (n_2+1) = n_3}{(n_1+1) \dot- ((n_2+1)+1) = n_3}\,[\text{minus}_2]\end{array} \quad \Rightarrow \quad \begin{array}{c}\mathcal{D}_2\\ \dfrac{n_1 \dot- n_2 = (n_3+1)}{(n_1+1) \dot- (n_2+1) = (n_3+1)}\,[\text{minus}_2]\end{array} \quad {}^{[\text{minus}\cdot\text{succ}]}}\,[\&2]$$

%mode $+\mathcal{D}_1 \quad \Rightarrow \quad -\mathcal{D}_2 \quad {}^{[\text{minus}\cdot\text{succ}]}$
%worlds $()$ $\mathcal{D}_1 \quad \Rightarrow \quad \mathcal{D}_2 \quad {}^{[\text{minus}\cdot\text{succ}]}$
%total $\mathcal{D}_1$ $\mathcal{D}_1 \quad \Rightarrow \quad \mathcal{D}_2 \quad {}^{[\text{minus}\cdot\text{succ}]}$ $\square$

%lemma $n_1 \dot- n_2 = n_3 \quad \Rightarrow \quad n_1 \dot- n_3 = n_2 \quad {}^{[\text{minus}\cdot\text{swap}]}$

Proof.

$$\frac{n_1\colon \textit{index}\ \cdot\ \begin{array}{c}\mathcal{D}\\ n_1 \dot- n_1 = 0\end{array} \quad {}^{[\text{minus}\cdot\text{id}]}}{n_1 \dot- 0 = n_1\;{}^{[\text{minus}_1]} \quad \Rightarrow \quad \begin{array}{c}\mathcal{D}\\ n_1 \dot- n_1 = 0\end{array} \quad {}^{[\text{minus}\cdot\text{swap}]}}\,[\&1]$$

$$\frac{\begin{array}{c}\begin{array}{c}\mathcal{D}_1\\ n_1 \dot- n_2 = n_3\end{array} \quad \Rightarrow \quad \begin{array}{c}\mathcal{D}_2\\ n_1 \dot- n_3 = n_2\end{array} \quad {}^{[\text{minus}\cdot\text{swap}]}\\ \begin{array}{c}\mathcal{D}_2\\ \dfrac{n_1 \dot- n_3 = n_2}{(n_1+1) \dot- (n_3+1) = n_2}\,[\text{minus}_2]\end{array} \quad \Rightarrow \quad \begin{array}{c}\mathcal{D}_3\\ (n_1+1) \dot- n_3 = (n_2+1)\end{array} \quad {}^{[\text{minus}\cdot\text{succ}]}\end{array}}{\begin{array}{c}\mathcal{D}_1\\ \dfrac{n_1 \dot- n_2 = n_3}{(n_1+1) \dot- (n_2+1) = n_3}\,[\text{minus}_2]\end{array} \quad \Rightarrow \quad \begin{array}{c}\mathcal{D}_3\\ (n_1+1) \dot- n_3 = (n_2+1)\end{array} \quad {}^{[\text{minus}\cdot\text{swap}]}}\,[\&2]$$

%mode $+\mathcal{D}_1 \quad \Rightarrow \quad -\mathcal{D}_2 \quad {}^{[\text{minus}\cdot\text{swap}]}$
%worlds $()$ $\mathcal{D}_1 \quad \Rightarrow \quad \mathcal{D}_2 \quad {}^{[\text{minus}\cdot\text{swap}]}$
%total $\mathcal{D}_1$ $\mathcal{D}_1 \quad \Rightarrow \quad \mathcal{D}_2 \quad {}^{[\text{minus}\cdot\text{swap}]}$ $\square$

1.2.1 Fetch (indices)

%judgment $\mathcal{I}(n_1) = n_2$

$$(n :: \mathcal{I})(0) = n \;{}^{[\text{fetch}_1^{\mathcal{I}}]}$$
$$(n :: \mathcal{I})(n_1+1) = n_2 \;{}^{[\text{fetch}_2^{\mathcal{I}}]} \quad \textit{when} \quad \mathcal{I}(n_1) = n_2$$

%mode $+\mathcal{I}(+n_1) = -n_2$
%worlds $()$ $\mathcal{I}(n_1) = n_2$
%terminates n_1 $\mathcal{I}(n_1) = n_2$
%unique $+\mathcal{I}(+n_1) = -1n_2$

1.2.2 Fetch (table)

%judgment $\mathcal{I}_\mu(n) = \mathcal{I}$

$$(\mathcal{I} :: \mathcal{I}_\mu)(0) = \mathcal{I}^{\left[\text{fetch}_1^{\mathcal{I}_\mu}\right]}$$
$$(\mathcal{I}' :: \mathcal{I}_\mu)(\alpha+1) = \mathcal{I}^{\left[\text{fetch}_2^{\mathcal{I}_\mu}\right]} \quad \textit{when} \quad \mathcal{I}_\mu(\alpha) = \mathcal{I}$$

%mode $+\mathcal{I}_\mu(+\alpha) = -\mathcal{I}$
%worlds $()$ $\mathcal{I}_\mu(\alpha) = \mathcal{I}$
%terminates α $\mathcal{I}_\mu(\alpha) = \mathcal{I}$
%unique $+\mathcal{I}_\mu(+\alpha) = -1\mathcal{I}$

1.2.3 Compute

%judgment $n_1 \dot- \mathcal{I}(n_2) = n_3$

$$n \dot- \mathcal{I}(l) = g \;{}^{[\text{compute}_1]} \quad \textit{when} \quad \mathcal{I}(l) = k, \quad n \dot- k = g$$

%mode $+n_1 \dot- +\mathcal{I}(+n_2) = -n_3$
%worlds $()$ $n_1 \dot- \mathcal{I}(n_2) = n_3$
%terminates $\{\}$ $n_1 \dot- \mathcal{I}(n_2) = n_3$
%unique $+n_1 \dot- +\mathcal{I}(+n_2) = -1n_3$

2 Safe $\boldsymbol{\lambda_{ct}}$-terms

2.1 Safety

%judgment $n \in \mathcal{I}$

$$n \in (n :: \mathcal{I}) \quad {}^{[\text{member}_1]}$$
$$n \in (n' :: \mathcal{I}) \quad {}^{[\text{member}_2]} \quad \textit{when} \quad n \in \mathcal{I}$$

%mode $+n \in +\mathcal{I}$
%worlds $()$ $n \in \mathcal{I}$
%terminates $\mathcal{I}$ $n \in \mathcal{I}$

%lemma $\mathcal{I}(n) = k \quad \Rightarrow \quad k \in \mathcal{I} \quad {}^{[\text{target}]}$

Proof.

$$\frac{}{(n :: \mathcal{I})(0) = n\;{}^{[\text{fetch}_1^{\mathcal{I}}]} \quad \Rightarrow \quad n \in (n :: \mathcal{I})\;{}^{[\text{member}_1]} \quad {}^{[\text{target}]}}\,[\&1]$$

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \mathcal{I}(n) = k\end{array} \quad \Rightarrow \quad \begin{array}{c}\mathcal{D}_2\\ k \in \mathcal{I}\end{array} \quad {}^{[\text{target}]}}{\begin{array}{c}\mathcal{D}_1\\ \dfrac{\mathcal{I}(n) = k}{(k' :: \mathcal{I})(n+1) = k}\,[\text{fetch}_2^{\mathcal{I}}]\end{array} \quad \Rightarrow \quad \begin{array}{c}\mathcal{D}_2\\ \dfrac{k \in \mathcal{I}}{k \in (k' :: \mathcal{I})}\,[\text{member}_2]\end{array} \quad {}^{[\text{target}]}}\,[\&2]$$

%mode $+\mathcal{D}_1 \quad \Rightarrow \quad -\mathcal{D}_2 \quad {}^{[\text{target}]}$
%worlds $()$ $\mathcal{D}_1 \quad \Rightarrow \quad \mathcal{D}_2 \quad {}^{[\text{target}]}$
%total $(\mathcal{D}_1)$ $\mathcal{D}_1 \quad \Rightarrow \quad \mathcal{D}_2 \quad {}^{[\text{target}]}$ $\square$

%judgment $\textit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(t)$

$$\begin{array}{llll}
\textit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(g) & {}^{[\text{safe}_1]} & \textit{when} & n \dot- g = k, \quad k \in \mathcal{I} \\
\textit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(t\,u) & {}^{[\text{safe}_2]} & \textit{when} & \textit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(t), \quad \textit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(u) \\
\textit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(\lambda t) & {}^{[\text{safe}_3]} & \textit{when} & \textit{Safe}_{n+1}^{(n+1 :: \mathcal{I}),\mathcal{I}_\mu}(t) \\
\textit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(\mathbf{catch}\, t) & {}^{[\text{safe}_4]} & \textit{when} & \textit{Safe}_n^{\mathcal{I},(\mathcal{I} :: \mathcal{I}_\mu)}(t) \\
\textit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(\mathbf{throw}\, \alpha\, t) & {}^{[\text{safe}_5]} & \textit{when} & \mathcal{I}_\mu(\alpha) = \mathcal{I}', \quad \textit{Safe}_n^{\mathcal{I}',\mathcal{I}_\mu}(t)
\end{array}$$

%mode $\textit{Safe}_{+n}^{+\mathcal{I},+\mathcal{I}_\mu}(+t)$
%worlds $()$ $\textit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(t)$
%terminates t $\textit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(t)$

2.2 From local indices to global indices

%judgment $\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t_1) = t_2$

$$\begin{array}{llll}
\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(l) = g & {}^{[\downarrow_1]} & \textit{when} & n \dot- \mathcal{I}(l) = g \\
\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t\,u) = t'\,u' & {}^{[\downarrow_2]} & \textit{when} & \downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t) = t', \quad \downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(u) = u' \\
\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(\lambda t) = \lambda t' & {}^{[\downarrow_3]} & \textit{when} & \downarrow_{n+1}^{(n+1 :: \mathcal{I}),\mathcal{I}_\mu}(t) = t' \\
\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(\mathbf{get\text{-}context}\, t) = \mathbf{catch}\, t' & {}^{[\downarrow_4]} & \textit{when} & \downarrow_n^{\mathcal{I},(\mathcal{I} :: \mathcal{I}_\mu)}(t) = t' \\
\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(\mathbf{set\text{-}context}\, \alpha\, t) = \mathbf{throw}\, \alpha\, t' & {}^{[\downarrow_5]} & \textit{when} & \mathcal{I}_\mu(\alpha) = \mathcal{I}', \quad \downarrow_n^{\mathcal{I}',\mathcal{I}_\mu}(t) = t'
\end{array}$$

%mode $\downarrow_{+n}^{+\mathcal{I},+\mathcal{I}_\mu}(+t) = -t'$
%worlds $()$ $\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t) = t'$
%terminates t $\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t) = t'$
%unique $\downarrow_{+n}^{+\mathcal{I},+\mathcal{I}_\mu}(+t) = -1t'$

%lemma $\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t)=t' \quad\Rightarrow\quad \mathit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(t')$ [↓·safe]

Proof.

$$\dfrac{\begin{array}{ccc}\dfrac{\mathcal{D}_{111}}{\mathcal{I}(l)=k} & \Rightarrow & \dfrac{\mathcal{D}_2}{k\in\mathcal{I}}\ [\text{target}]\\ \dfrac{\mathcal{D}_{112}}{n\dot{-}k=g} & \Rightarrow & \dfrac{\mathcal{D}_3}{n\dot{-}g=k}\ [\text{minus-swap}]\end{array}}{\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_{111} & \mathcal{D}_{112}\\ \mathcal{I}(l)=k & n\dot{-}k=g\end{array}}{n\dot{-}\mathcal{I}(l)=g}\,[\text{compute}_1]}{\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(l)=g}\,[\downarrow_1] \quad\Rightarrow\quad \dfrac{\begin{array}{cc}\mathcal{D}_3 & \mathcal{D}_2\\ n\dot{-}g=k & k\in\mathcal{I}\end{array}}{\mathit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(g)}\,[\text{safe}_1]\ \ [\downarrow\cdot\text{safe}]}\,[\&1]$$

$$\dfrac{\begin{array}{ccc}\dfrac{\mathcal{D}_{11}}{\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t)=t'} & \Rightarrow & \dfrac{\mathcal{D}_1}{\mathit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(t')}\ [\downarrow\cdot\text{safe}]\\ \dfrac{\mathcal{D}_{12}}{\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(u)=u'} & \Rightarrow & \dfrac{\mathcal{D}_2}{\mathit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(u')}\ [\downarrow\cdot\text{safe}]\end{array}}{\dfrac{\begin{array}{cc}\mathcal{D}_{11} & \mathcal{D}_{12}\\ \downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t)=t' & \downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(u)=u'\end{array}}{\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t\,u)=t'\,u'}\,[\downarrow_2] \quad\Rightarrow\quad \dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \mathit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(t') & \mathit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(u')\end{array}}{\mathit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(t'\,u')}\,[\text{safe}_2]\ \ [\downarrow\cdot\text{safe}]}\,[\&2]$$

$$\dfrac{\dfrac{\mathcal{D}_{11}}{\downarrow_{n+1}^{(n+1::\mathcal{I}),\mathcal{I}_\mu}(t)=t'} \quad\Rightarrow\quad \dfrac{\mathcal{D}_1}{\mathit{Safe}_{n+1}^{(n+1::\mathcal{I}),\mathcal{I}_\mu}(t')}\ [\downarrow\cdot\text{safe}]}{\dfrac{\dfrac{\mathcal{D}_{11}}{\downarrow_{n+1}^{(n+1::\mathcal{I}),\mathcal{I}_\mu}(t)=t'}}{\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(\lambda t)=\lambda t'}\,[\downarrow_3] \quad\Rightarrow\quad \dfrac{\dfrac{\mathcal{D}_1}{\mathit{Safe}_{n+1}^{(n+1::\mathcal{I}),\mathcal{I}_\mu}(t')}}{\mathit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(\lambda t')}\,[\text{safe}_3]\ \ [\downarrow\cdot\text{safe}]}\,[\&3]$$

$$\dfrac{\dfrac{\mathcal{D}_{11}}{\downarrow_n^{\mathcal{I},(\mathcal{I}::\mathcal{I}_\mu)}(t)=t'} \quad\Rightarrow\quad \dfrac{\mathcal{D}_1}{\mathit{Safe}_n^{\mathcal{I},(\mathcal{I}::\mathcal{I}_\mu)}(t')}\ [\downarrow\cdot\text{safe}]}{\dfrac{\dfrac{\mathcal{D}_{11}}{\downarrow_n^{\mathcal{I},(\mathcal{I}::\mathcal{I}_\mu)}(t)=t'}}{\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(\textbf{get-context}\,t)=\textbf{catch}\,t'}\,[\downarrow_4] \quad\Rightarrow\quad \dfrac{\dfrac{\mathcal{D}_1}{\mathit{Safe}_n^{\mathcal{I},(\mathcal{I}::\mathcal{I}_\mu)}(t')}}{\mathit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(\textbf{catch}\,t')}\,[\text{safe}_4]\ \ [\downarrow\cdot\text{safe}]}\,[\&4]$$

$$\dfrac{\dfrac{\mathcal{D}_{12}}{\downarrow_n^{\mathcal{I}',\mathcal{I}_\mu}(t)=t'} \quad\Rightarrow\quad \dfrac{\mathcal{D}_2}{\mathit{Safe}_n^{\mathcal{I}',\mathcal{I}_\mu}(t')}\ [\downarrow\cdot\text{safe}]}{\dfrac{\begin{array}{cc}\mathcal{D}_{11} & \mathcal{D}_{12}\\ \mathcal{I}_\mu(\alpha)=\mathcal{I}' & \downarrow_n^{\mathcal{I}',\mathcal{I}_\mu}(t)=t'\end{array}}{\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(\textbf{set-context}\,\alpha\,t)=\textbf{throw}\,\alpha\,t'}\,[\downarrow_5] \quad\Rightarrow\quad \dfrac{\begin{array}{cc}\mathcal{D}_{11} & \mathcal{D}_2\\ \mathcal{I}_\mu(\alpha)=\mathcal{I}' & \mathit{Safe}_n^{\mathcal{I}',\mathcal{I}_\mu}(t')\end{array}}{\mathit{Safe}_n^{\mathcal{I},\mathcal{I}_\mu}(\textbf{throw}\,\alpha\,t')}\,[\text{safe}_5]\ \ [\downarrow\cdot\text{safe}]}\,[\&5]$$

%mode $+\mathcal{D}_1 \;\Rightarrow\; -\mathcal{D}_2$ [↓·safe]
%worlds $()\;\; \mathcal{D}_1 \;\Rightarrow\; \mathcal{D}_2$ [↓·safe]
%total $(\mathcal{D}_1)\;\; \mathcal{D}_1 \;\Rightarrow\; \mathcal{D}_2$ [↓·safe]

□

2.3 Examples

$1 \;=\; 0+1.$
$2 \;=\; 1+1.$
$3 \;=\; 2+1.$

%solve _ : $\mathit{Safe}_0^{[],[]}(\lambda\textbf{catch}\,\lambda(0\,(\textbf{throw}\,0\,1)))$

Remark. This example fails as expected:
%solve _ : $\mathit{Safe}_0^{[],[]}(\lambda\textbf{catch}\,\lambda(1\,(\textbf{throw}\,0\,0)))$

%solve _ : $\downarrow_0^{[],[]}(\lambda\textbf{get-context}\,\lambda(1\,(\textbf{set-context}\,0\,0)))=\lambda\textbf{catch}\,\lambda(1\,(\textbf{throw}\,0\,1))$

%solve d_1 : $\downarrow_0^{[],[]}(\lambda\textbf{get-context}\,\lambda(1\,(\textbf{set-context}\,0\,0)))=\lambda\textbf{catch}\,\lambda(1\,(\textbf{throw}\,0\,1))$
%solve _ : $d_1 \;\Rightarrow\; \mathcal{D}_2$ [↓·safe]

2.4 Closure, environment and stack

%datatype *clos* **%name** *clos* c
%datatype *c-env* **%name** *c-env* $\mathcal{E}$
%datatype *k-env* **%name** *k-env* $\mathcal{E}_\mu$
%datatype *stack* **%name** *stack* $\mathcal{S}$

$$\begin{array}{lcl} c & ::= & (t,\mathcal{E},\mathcal{E}_\mu)\\[1ex] \mathcal{E} & ::= & ()\\ & \mid & (c;\mathcal{E})\\[1ex] \mathcal{E}_\mu & ::= & ()\\ & \mid & (\mathcal{S};\mathcal{E}_\mu)\\[1ex] \mathcal{S} & ::= & []\\ & \mid & c::\mathcal{S}\end{array}$$

%datatype *state*
%name *state* σ

$$\sigma \;::=\; \langle t,\mathcal{E},\mathcal{E}_\mu,\mathcal{S}\rangle$$

2.5 Judgments

2.5.1 Fetch a closure

%judgment $\mathcal{E}(n)=c$

$$\begin{array}{l}(c;\mathcal{E})(0)=c\ ^{[\text{fetch}_1]}\\ (c';\mathcal{E})(n+1)=c\ ^{[\text{fetch}_2]}\quad \mathit{when}\quad \mathcal{E}(n)=c\end{array}$$

%mode $+\mathcal{E}(+n)=-c$
%worlds $()\;\; \mathcal{E}(n)=c$
%terminates $\mathcal{E}\;\; \mathcal{E}(n)=c$
%unique $+\mathcal{E}(+n)=-1c$

2.5.2 Fetch a stack

%judgment $\mathcal{E}_\mu(n)=\mathcal{S}$

$$\begin{array}{l}(\mathcal{S};\mathcal{E}_\mu)(0)=\mathcal{S}\ ^{[\text{fetch}_1^\mu]}\\ (\mathcal{S}';\mathcal{E}_\mu)(n+1)=\mathcal{S}\ ^{[\text{fetch}_2^\mu]}\quad \mathit{when}\quad \mathcal{E}_\mu(n)=\mathcal{S}\end{array}$$

%mode $+\mathcal{E}_\mu(+n)=-\mathcal{S}$
%worlds $()\;\; \mathcal{E}_\mu(n)=\mathcal{S}$
%terminates $\mathcal{E}_\mu\;\; \mathcal{E}_\mu(n)=\mathcal{S}$
%unique $+\mathcal{E}_\mu(+n)=-1\mathcal{S}$

2.5.3 Evaluation rules

%judgment $\sigma_1 \rightsquigarrow \sigma_2$

$$\begin{array}{l}\langle k,\mathcal{E},\mathcal{E}_\mu,\mathcal{S}\rangle \rightsquigarrow \langle t,\mathcal{E}',\mathcal{E}'_\mu,\mathcal{S}\rangle\ ^{[\text{k}\cdot\text{var}]}\quad \mathit{when}\quad \mathcal{E}(k)=(t,\mathcal{E}',\mathcal{E}'_\mu)\\ \langle (t\,u),\mathcal{E},\mathcal{E}_\mu,\mathcal{S}\rangle \rightsquigarrow \langle t,\mathcal{E},\mathcal{E}_\mu,(u,\mathcal{E},\mathcal{E}_\mu)::\mathcal{S}\rangle\ ^{[\text{k}\cdot\text{app}]}\\ \langle \lambda t,\mathcal{E},\mathcal{E}_\mu,c::\mathcal{S}\rangle \rightsquigarrow \langle t,(c;\mathcal{E}),\mathcal{E}_\mu,\mathcal{S}\rangle\ ^{[\text{k}\cdot\text{abs}]}\\ \langle \textbf{catch}\,t,\mathcal{E},\mathcal{E}_\mu,\mathcal{S}\rangle \rightsquigarrow \langle t,\mathcal{E},(\mathcal{S};\mathcal{E}_\mu),\mathcal{S}\rangle\ ^{[\text{k}\cdot\text{catch}]}\\ \langle \textbf{throw}\,\alpha\,t,\mathcal{E},\mathcal{E}_\mu,\mathcal{S}\rangle \rightsquigarrow \langle t,\mathcal{E},\mathcal{E}_\mu,\mathcal{S}'\rangle\ ^{[\text{k}\cdot\text{throw}]}\quad \mathit{when}\quad \mathcal{E}_\mu(\alpha)=\mathcal{S}'\end{array}$$

%mode $+\sigma_1 \rightsquigarrow -\sigma_2$
%worlds $()\;\; \sigma_1 \rightsquigarrow \sigma_2$
%unique $+\sigma_1 \rightsquigarrow -1\sigma_2$

2.6 Abstract machine for safe $\boldsymbol{\lambda_{\text{ct}}}$-terms (with indirection tables)

2.6.1 Syntax

%datatype *clos*
%datatype *c-env*
%datatype *k-env*
%datatype *stack*
%name *clos* $\tilde{c}$
%name *c-env* $\tilde{\mathcal{E}}$
%name *k-env* $\tilde{\mathcal{E}}_\mu$
%name *stack* $\tilde{\mathcal{S}}$

$$\begin{array}{lcl} \tilde{c} & ::= & (t,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu)\\[1ex] \tilde{\mathcal{E}} & ::= & ()\\ & \mid & (\tilde{c};\tilde{\mathcal{E}})\\[1ex] \tilde{\mathcal{E}}_\mu & ::= & ()\\ & \mid & (\tilde{\mathcal{S}};\tilde{\mathcal{E}}_\mu)\\[1ex] \tilde{\mathcal{S}} & ::= & []\\ & \mid & \tilde{c}::\tilde{\mathcal{S}}\end{array}$$

%datatype *state*
%name *state* $\tilde{\sigma}$

$$\tilde{\sigma} \;::=\; \langle t,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu,\tilde{\mathcal{S}}\rangle$$

2.6.2 Fetch a closure

%judgment $\tilde{\mathcal{E}}(n)=\tilde{c}$

$$\begin{array}{l}(\tilde{c};\tilde{\mathcal{E}})(0)=\tilde{c}\ ^{[\text{i}\cdot\text{fetch}_1]}\\ (\tilde{c}';\tilde{\mathcal{E}})(n+1)=\tilde{c}\ ^{[\text{i}\cdot\text{fetch}_2]}\quad \mathit{when}\quad \tilde{\mathcal{E}}(n)=\tilde{c}\end{array}$$

%mode $+\tilde{\mathcal{E}}(+n)=-\tilde{c}$
%worlds $()\;\; \tilde{\mathcal{E}}(n)=\tilde{c}$
%terminates $\tilde{\mathcal{E}}\;\; \tilde{\mathcal{E}}(n)=\tilde{c}$
%unique $+\tilde{\mathcal{E}}(+n)=-1\tilde{c}$

2.6.3 Fetch a stack

%judgment $\tilde{\mathcal{E}}_\mu(n)=\tilde{\mathcal{S}}$

$$\begin{array}{l}(\tilde{\mathcal{S}};\tilde{\mathcal{E}}_\mu)(0)=\tilde{\mathcal{S}}\ ^{[\text{i}\cdot\text{fetch}_1^\mu]}\\ (\tilde{\mathcal{S}}';\tilde{\mathcal{E}}_\mu)(n+1)=\tilde{\mathcal{S}}\ ^{[\text{i}\cdot\text{fetch}_2^\mu]}\quad \mathit{when}\quad \tilde{\mathcal{E}}_\mu(n)=\tilde{\mathcal{S}}\end{array}$$

%mode $+\tilde{\mathcal{E}}_\mu(+n)=-\tilde{\mathcal{S}}$
%worlds $()\;\; \tilde{\mathcal{E}}_\mu(n)=\tilde{\mathcal{S}}$
%terminates $\tilde{\mathcal{E}}_\mu\;\; \tilde{\mathcal{E}}_\mu(n)=\tilde{\mathcal{S}}$
%unique $+\tilde{\mathcal{E}}_\mu(+n)=-1\tilde{\mathcal{S}}$

3 Evaluation rules

%judgment $\tilde{\sigma}_1 \rightsquigarrow \tilde{\sigma}_2$

$$\begin{array}{l}\langle l,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu,\tilde{\mathcal{S}}\rangle \rightsquigarrow \langle t,n',\mathcal{I}',\mathcal{I}'_\mu,\tilde{\mathcal{E}}',\tilde{\mathcal{E}}'_\mu,\tilde{\mathcal{S}}\rangle\ ^{[\text{i}\cdot\text{var}]}\\ \qquad \mathit{when}\quad n\dot{-}\mathcal{I}(l)=g,\quad \tilde{\mathcal{E}}(g)=\big(t,n',\mathcal{I}',\mathcal{I}'_\mu,\tilde{\mathcal{E}}',\tilde{\mathcal{E}}'_\mu\big)\\ \langle (t\,u),n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu,\tilde{\mathcal{S}}\rangle \rightsquigarrow \langle t,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu,\big(u,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu\big)::\tilde{\mathcal{S}}\rangle\ ^{[\text{i}\cdot\text{app}]}\\ \langle \lambda t,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu,\tilde{c}::\tilde{\mathcal{S}}\rangle \rightsquigarrow \langle t,n+1,(n+1::\mathcal{I}),\mathcal{I}_\mu,(\tilde{c};\tilde{\mathcal{E}}),\tilde{\mathcal{E}}_\mu,\tilde{\mathcal{S}}\rangle\ ^{[\text{i}\cdot\text{abs}]}\\ \langle \textbf{get-context}\,t,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu,\tilde{\mathcal{S}}\rangle \rightsquigarrow \langle t,n,\mathcal{I},(\mathcal{I}::\mathcal{I}_\mu),\tilde{\mathcal{E}},\big(\tilde{\mathcal{S}};\tilde{\mathcal{E}}_\mu\big),\tilde{\mathcal{S}}\rangle\ ^{[\text{i}\cdot\text{catch}]}\\ \langle \textbf{set-context}\,\alpha\,t,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu,\tilde{\mathcal{S}}\rangle \rightsquigarrow \langle t,n,\mathcal{I}',\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu,\tilde{\mathcal{S}}'\rangle\ ^{[\text{i}\cdot\text{throw}]}\\ \qquad \mathit{when}\quad \mathcal{I}_\mu(\alpha)=\mathcal{I}',\quad \tilde{\mathcal{E}}_\mu(\alpha)=\tilde{\mathcal{S}}'\end{array}$$

%mode $+\tilde{\sigma}_1 \rightsquigarrow -\tilde{\sigma}_2$
%worlds $()\;\; \tilde{\sigma}_1 \rightsquigarrow \tilde{\sigma}_2$
%unique $+\tilde{\sigma}_1 \rightsquigarrow -1\tilde{\sigma}_2$

4 Translation

%judgment $\tilde{c}^* = c$
%judgment $\tilde{\mathcal{S}}^* = \mathcal{S}$
%judgment $\tilde{\mathcal{E}}^* = \mathcal{E}$
%judgment $\tilde{\mathcal{E}}_\mu{}^* = \mathcal{E}_\mu$

$$\big(t,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu\big)^* = (u,\mathcal{E},\mathcal{E}_\mu)\ ^{[\text{clos}^*]}\quad \mathit{when}\quad \downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t)=u,\quad \tilde{\mathcal{E}}^*=\mathcal{E},\quad \tilde{\mathcal{E}}_\mu{}^*=\mathcal{E}_\mu$$

$[]^\star = []\ ^{[\text{stack}^\star_1]}$
$(\tilde c :: \tilde{\mathcal{S}})^\star = c :: \mathcal{S}\ ^{[\text{stack}^\star_2]}\quad \textit{when}\quad \tilde c^\star = c,\quad \tilde{\mathcal{S}}^\star = \mathcal{S}$

$()^\star = ()\ ^{[\text{c·env}^\star_1]}$
$(\tilde c; \tilde{\mathcal{E}})^\star = (c; \mathcal{E})\ ^{[\text{c·env}^\star_2]}\quad \textit{when}\quad \tilde c^\star = c,\quad \tilde{\mathcal{E}}^\star = \mathcal{E}$

$()^\star = ()\ ^{[\text{k·env}^\star_1]}$
$(\tilde{\mathcal{S}}; \tilde{\mathcal{E}}_\mu)^\star = (\mathcal{S}; \mathcal{E}_\mu)\ ^{[\text{k·env}^\star_2]}\quad \textit{when}\quad \tilde{\mathcal{S}}^\star = \mathcal{S},\quad \tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu$

%mode
$+\tilde c^\star = -c$
$+\tilde{\mathcal{S}}^\star = -\mathcal{S}$
$+\tilde{\mathcal{E}}^\star = -\mathcal{E}$
$+\tilde{\mathcal{E}}_\mu{}^\star = -\mathcal{E}_\mu$

%worlds $()$
$\tilde c^\star = c$
$\tilde{\mathcal{S}}^\star = \mathcal{S}$
$\tilde{\mathcal{E}}^\star = \mathcal{E}$
$\tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu$

%terminates $(\tilde c\ \tilde{\mathcal{S}}\ \tilde{\mathcal{E}}\ \tilde{\mathcal{E}}_\mu)$
$\tilde c^\star = c$
$\tilde{\mathcal{S}}^\star = \mathcal{S}$
$\tilde{\mathcal{E}}^\star = \mathcal{E}$
$\tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu$

%unique
$+\tilde c^\star = -1c$
$+\tilde{\mathcal{S}}^\star = -1\mathcal{S}$
$+\tilde{\mathcal{E}}^\star = -1\mathcal{E}$
$+\tilde{\mathcal{E}}_\mu{}^\star = -1\mathcal{E}_\mu$

%judgment $\tilde\sigma^\star = \sigma$
$\langle t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, \tilde{\mathcal{S}}\rangle^\star = \langle u, \mathcal{E}, \mathcal{E}_\mu, \mathcal{S}\rangle\ ^{[\text{state}^\star]}\quad \textit{when}\quad (t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu)^\star = (u, \mathcal{E}, \mathcal{E}_\mu),\quad \tilde{\mathcal{S}}^\star = \mathcal{S}$

%mode $+\tilde\sigma^\star = -\sigma$
%worlds $()\quad \tilde\sigma^\star = \sigma$
%unique $+\tilde\sigma^\star = -1\sigma$

5 Soundness

%lemma $\tilde{\mathcal{E}}^\star = \mathcal{E}\ \wedge\ \tilde{\mathcal{E}}(n) = \tilde c\ \Rightarrow\ \tilde c^\star = c\ \wedge\ \mathcal{E}(n) = c\ \ \textit{for some } c\ \ ^{[\text{fetch·sound}]}$

Proof.

$$\dfrac{}{\dfrac{\begin{matrix}\mathcal{D}_{11}\\ \tilde c^\star = c\end{matrix}\quad \begin{matrix}\vdots\\ \tilde{\mathcal{E}}^\star = \mathcal{E}\end{matrix}}{(\tilde c; \tilde{\mathcal{E}})^\star = (c; \mathcal{E})}\,[\text{c·env}^\star_2]\ \wedge\ (\tilde c; \tilde{\mathcal{E}})(0) = \tilde c\ ^{[\text{i·fetch}_1]}\ \Rightarrow\ \begin{matrix}\mathcal{D}_{11}\\ \tilde c^\star = c\end{matrix}\ \wedge\ (c; \mathcal{E})(0) = c\ ^{[\text{fetch}_1]}\quad ^{[\text{fetch·sound}]}}\,[\&1]$$

$$\dfrac{\begin{matrix}\mathcal{D}_{12}\\ \tilde{\mathcal{E}}^\star = \mathcal{E}\end{matrix}\ \wedge\ \begin{matrix}\mathcal{D}_{2}\\ \tilde{\mathcal{E}}(n) = \tilde c\end{matrix}\ \Rightarrow\ \begin{matrix}\mathcal{D}_{3}\\ \tilde c^\star = c\end{matrix}\ \wedge\ \begin{matrix}\mathcal{D}_{4}\\ \mathcal{E}(n) = c\end{matrix}\quad ^{[\text{fetch·sound}]}}{\dfrac{\begin{matrix}\vdots\\ \tilde c'^\star = c'\end{matrix}\quad \begin{matrix}\mathcal{D}_{12}\\ \tilde{\mathcal{E}}^\star = \mathcal{E}\end{matrix}}{(\tilde c'; \tilde{\mathcal{E}})^\star = (c'; \mathcal{E})}\,[\text{c·env}^\star_2]\ \wedge\ \dfrac{\begin{matrix}\mathcal{D}_{2}\\ \tilde{\mathcal{E}}(n) = \tilde c\end{matrix}}{(\tilde c'; \tilde{\mathcal{E}})(n+1) = \tilde c}\,[\text{i·fetch}_2]\ \Rightarrow\ \begin{matrix}\mathcal{D}_{3}\\ \tilde c^\star = c\end{matrix}\ \wedge\ \dfrac{\begin{matrix}\mathcal{D}_{4}\\ \mathcal{E}(n) = c\end{matrix}}{(c'; \mathcal{E})(n+1) = c}\,[\text{fetch}_2]\quad ^{[\text{fetch·sound}]}}\,[\&2]$$

%mode $+\mathcal{D}_1\ \wedge\ +\mathcal{D}_2\ \Rightarrow\ -\mathcal{D}_3\ \wedge\ -\mathcal{D}_4\ ^{[\text{fetch·sound}]}$
%worlds $()\ \ \mathcal{D}_1\ \wedge\ \mathcal{D}_2\ \Rightarrow\ \mathcal{D}_3\ \wedge\ \mathcal{D}_4\ ^{[\text{fetch·sound}]}$
%total $(\mathcal{D}_2)\ \ \mathcal{D}_1\ \wedge\ \mathcal{D}_2\ \Rightarrow\ \mathcal{D}_3\ \wedge\ \mathcal{D}_4\ ^{[\text{fetch·sound}]}$ □

%lemma $\tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu\ \wedge\ \tilde{\mathcal{E}}_\mu(\alpha) = \tilde{\mathcal{S}}\ \Rightarrow\ \tilde{\mathcal{S}}^\star = \mathcal{S}\ \wedge\ \mathcal{E}_\mu(\alpha) = \mathcal{S}\ \ \textit{for some } \mathcal{S}\ \ ^{[\text{fetch}^\mu\text{·sound}]}$

Proof.

$$\dfrac{}{\dfrac{\begin{matrix}\mathcal{D}_{11}\\ \tilde{\mathcal{S}}^\star = \mathcal{S}\end{matrix}\quad \begin{matrix}\vdots\\ \tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu\end{matrix}}{(\tilde{\mathcal{S}}; \tilde{\mathcal{E}}_\mu)^\star = (\mathcal{S}; \mathcal{E}_\mu)}\,[\text{k·env}^\star_2]\ \wedge\ (\tilde{\mathcal{S}}; \tilde{\mathcal{E}}_\mu)(0) = \tilde{\mathcal{S}}\ ^{[\text{i·fetch}^\mu_1]}\ \Rightarrow\ \begin{matrix}\mathcal{D}_{11}\\ \tilde{\mathcal{S}}^\star = \mathcal{S}\end{matrix}\ \wedge\ (\mathcal{S}; \mathcal{E}_\mu)(0) = \mathcal{S}\ ^{[\text{fetch}^\mu_1]}\quad ^{[\text{fetch}^\mu\text{·sound}]}}\,[\&1]$$

$$\dfrac{\begin{matrix}\mathcal{D}_{12}\\ \tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu\end{matrix}\ \wedge\ \begin{matrix}\mathcal{D}_{2}\\ \tilde{\mathcal{E}}_\mu(\alpha) = \tilde{\mathcal{S}}\end{matrix}\ \Rightarrow\ \begin{matrix}\mathcal{D}_{3}\\ \tilde{\mathcal{S}}^\star = \mathcal{S}\end{matrix}\ \wedge\ \begin{matrix}\mathcal{D}_{4}\\ \mathcal{E}_\mu(\alpha) = \mathcal{S}\end{matrix}\quad ^{[\text{fetch}^\mu\text{·sound}]}}{\dfrac{\begin{matrix}\vdots\\ \tilde{\mathcal{S}}'^\star = \mathcal{S}'\end{matrix}\quad \begin{matrix}\mathcal{D}_{12}\\ \tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu\end{matrix}}{(\tilde{\mathcal{S}}'; \tilde{\mathcal{E}}_\mu)^\star = (\mathcal{S}'; \mathcal{E}_\mu)}\,[\text{k·env}^\star_2]\ \wedge\ \dfrac{\begin{matrix}\mathcal{D}_{2}\\ \tilde{\mathcal{E}}_\mu(\alpha) = \tilde{\mathcal{S}}\end{matrix}}{(\tilde{\mathcal{S}}'; \tilde{\mathcal{E}}_\mu)(\alpha+1) = \tilde{\mathcal{S}}}\,[\text{i·fetch}^\mu_2]\ \Rightarrow\ \begin{matrix}\mathcal{D}_{3}\\ \tilde{\mathcal{S}}^\star = \mathcal{S}\end{matrix}\ \wedge\ \dfrac{\begin{matrix}\mathcal{D}_{4}\\ \mathcal{E}_\mu(\alpha) = \mathcal{S}\end{matrix}}{(\mathcal{S}'; \mathcal{E}_\mu)(\alpha+1) = \mathcal{S}}\,[\text{fetch}^\mu_2]\quad ^{[\text{fetch}^\mu\text{·sound}]}}\,[\&2]$$

%mode $+\mathcal{D}_1\ \wedge\ +\mathcal{D}_2\ \Rightarrow\ -\mathcal{D}_3\ \wedge\ -\mathcal{D}_4\ ^{[\text{fetch}^\mu\text{·sound}]}$
%worlds $()\ \ \mathcal{D}_1\ \wedge\ \mathcal{D}_2\ \Rightarrow\ \mathcal{D}_3\ \wedge\ \mathcal{D}_4\ ^{[\text{fetch}^\mu\text{·sound}]}$
%total $(\mathcal{D}_2)\ \ \mathcal{D}_1\ \wedge\ \mathcal{D}_2\ \Rightarrow\ \mathcal{D}_3\ \wedge\ \mathcal{D}_4\ ^{[\text{fetch}^\mu\text{·sound}]}$ □

%theorem $\tilde\sigma_1 \rightsquigarrow \tilde\sigma_2\ \wedge\ \tilde\sigma_1{}^\star = \sigma_1\ \Rightarrow\ \sigma_1 \rightsquigarrow \sigma_2\ \wedge\ \tilde\sigma_2{}^\star = \sigma_2\ \ \textit{for some } \sigma_2\ \ ^{[\text{soundness}]}$

Proof.

$$\dfrac{\begin{matrix}\mathcal{D}_{212}\\ \tilde{\mathcal{E}}^\star = \mathcal{E}\end{matrix}\ \wedge\ \begin{matrix}\mathcal{D}_{13}\\ \tilde{\mathcal{E}}(g) = (t, n', \mathcal{I}', \mathcal{I}'_\mu, \tilde{\mathcal{E}}', \tilde{\mathcal{E}}'_\mu)\end{matrix}\ \Rightarrow\ \begin{matrix}\mathcal{D}_{31}\\ (t, n', \mathcal{I}', \mathcal{I}'_\mu, \tilde{\mathcal{E}}', \tilde{\mathcal{E}}'_\mu)^\star = (u, \mathcal{E}', \mathcal{E}'_\mu)\end{matrix}\ \wedge\ \begin{matrix}\mathcal{D}_{4}\\ \mathcal{E}(g) = (u, \mathcal{E}', \mathcal{E}'_\mu)\end{matrix}\quad ^{[\text{fetch·sound}]}}{\dfrac{\begin{matrix}\mathcal{D}_{11}\\ n \dot- \mathcal{I}(l) = g\end{matrix}\quad \begin{matrix}\mathcal{D}_{13}\\ \tilde{\mathcal{E}}(g) = (t, n', \mathcal{I}', \mathcal{I}'_\mu, \tilde{\mathcal{E}}', \tilde{\mathcal{E}}'_\mu)\end{matrix}}{\langle l, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, \tilde{\mathcal{S}}\rangle \rightsquigarrow \langle t, n', \mathcal{I}', \mathcal{I}'_\mu, \tilde{\mathcal{E}}', \tilde{\mathcal{E}}'_\mu, \tilde{\mathcal{S}}\rangle}\,[\text{i·var}]\ \wedge\ \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ n \dot- \mathcal{I}(l) = g\end{matrix}}{\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(l) = g}\,[\downarrow_1]\quad \begin{matrix}\mathcal{D}_{212}\\ \tilde{\mathcal{E}}^\star = \mathcal{E}\end{matrix}\quad \begin{matrix}\vdots\\ \tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu\end{matrix}}{(l, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu)^\star = (g, \mathcal{E}, \mathcal{E}_\mu)}\,[\text{clos}^\star]\quad \begin{matrix}\mathcal{D}_{22}\\ \tilde{\mathcal{S}}^\star = \mathcal{S}\end{matrix}}{\langle l, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, \tilde{\mathcal{S}}\rangle^\star = \langle g, \mathcal{E}, \mathcal{E}_\mu, \mathcal{S}\rangle}\,[\text{state}^\star]\ \Rightarrow\ \dfrac{\begin{matrix}\mathcal{D}_{4}\\ \mathcal{E}(g) = (u, \mathcal{E}', \mathcal{E}'_\mu)\end{matrix}}{\langle g, \mathcal{E}, \mathcal{E}_\mu, \mathcal{S}\rangle \rightsquigarrow \langle u, \mathcal{E}', \mathcal{E}'_\mu, \mathcal{S}\rangle}\,[\text{k·var}]\ \wedge\ \dfrac{\begin{matrix}\mathcal{D}_{31}\\ (t, n', \mathcal{I}', \mathcal{I}'_\mu, \tilde{\mathcal{E}}', \tilde{\mathcal{E}}'_\mu)^\star = (u, \mathcal{E}', \mathcal{E}'_\mu)\end{matrix}\quad \begin{matrix}\mathcal{D}_{22}\\ \tilde{\mathcal{S}}^\star = \mathcal{S}\end{matrix}}{\langle t, n', \mathcal{I}', \mathcal{I}'_\mu, \tilde{\mathcal{E}}', \tilde{\mathcal{E}}'_\mu, \tilde{\mathcal{S}}\rangle^\star = \langle u, \mathcal{E}', \mathcal{E}'_\mu, \mathcal{S}\rangle}\,[\text{state}^\star]\quad ^{[\text{soundness}]}}\,[\&1]$$

$$\dfrac{}{\langle (t\,u), n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, \tilde{\mathcal{S}}\rangle \rightsquigarrow \langle t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, (u, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu) :: \tilde{\mathcal{S}}\rangle\ ^{[\text{i·app}]}\ \wedge\ \dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_{2111}\\ \downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t) = t'\end{matrix}\quad \begin{matrix}\mathcal{D}_{2112}\\ \downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(u) = u'\end{matrix}}{\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t\,u) = (t'\,u')}\,[\downarrow_2]\quad \begin{matrix}\mathcal{D}_{212}\\ \tilde{\mathcal{E}}^\star = \mathcal{E}\end{matrix}\quad \begin{matrix}\mathcal{D}_{213}\\ \tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu\end{matrix}}{((t\,u), n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu)^\star = ((t'\,u'), \mathcal{E}, \mathcal{E}_\mu)}\,[\text{clos}^\star]\quad \begin{matrix}\mathcal{D}_{23}\\ \tilde{\mathcal{S}}^\star = \mathcal{S}\end{matrix}}{\langle (t\,u), n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, \tilde{\mathcal{S}}\rangle^\star = \langle (t'\,u'), \mathcal{E}, \mathcal{E}_\mu, \mathcal{S}\rangle}\,[\text{state}^\star]\ \Rightarrow\ \langle (t'\,u'), \mathcal{E}, \mathcal{E}_\mu, \mathcal{S}\rangle \rightsquigarrow \langle t', \mathcal{E}, \mathcal{E}_\mu, (u', \mathcal{E}, \mathcal{E}_\mu) :: \mathcal{S}\rangle\ ^{[\text{k·app}]}\ \wedge\ \dfrac{\dfrac{\begin{matrix}\mathcal{D}_{2111}\\ \downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t) = t'\end{matrix}\quad \begin{matrix}\mathcal{D}_{212}\\ \tilde{\mathcal{E}}^\star = \mathcal{E}\end{matrix}\quad \begin{matrix}\mathcal{D}_{213}\\ \tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu\end{matrix}}{(t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu)^\star = (t', \mathcal{E}, \mathcal{E}_\mu)}\,[\text{clos}^\star]\quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_{2112}\\ \downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(u) = u'\end{matrix}\quad \begin{matrix}\mathcal{D}_{212}\\ \tilde{\mathcal{E}}^\star = \mathcal{E}\end{matrix}\quad \begin{matrix}\mathcal{D}_{213}\\ \tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu\end{matrix}}{(u, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu)^\star = (u', \mathcal{E}, \mathcal{E}_\mu)}\,[\text{clos}^\star]\quad \begin{matrix}\mathcal{D}_{23}\\ \tilde{\mathcal{S}}^\star = \mathcal{S}\end{matrix}}{((u, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu) :: \tilde{\mathcal{S}})^\star = (u', \mathcal{E}, \mathcal{E}_\mu) :: \mathcal{S}}\,[\text{stack}^\star_2]}{\langle t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, (u, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu) :: \tilde{\mathcal{S}}\rangle^\star = \langle t', \mathcal{E}, \mathcal{E}_\mu, (u', \mathcal{E}, \mathcal{E}_\mu) :: \mathcal{S}\rangle}\,[\text{state}^\star]\quad ^{[\text{soundness}]}}\,[\&2]$$

$$\dfrac{}{\langle \lambda t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, \tilde c :: \tilde{\mathcal{S}}\rangle \rightsquigarrow \langle t, n+1, (n+1 :: \mathcal{I}), \mathcal{I}_\mu, (\tilde c; \tilde{\mathcal{E}}), \tilde{\mathcal{E}}_\mu, \tilde{\mathcal{S}}\rangle\ ^{[\text{i·abs}]}\ \wedge\ \dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_{2111}\\ \downarrow_{n+1}^{(n+1 :: \mathcal{I}),\mathcal{I}_\mu}(t) = t'\end{matrix}}{\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(\lambda t) = \lambda t'}\,[\downarrow_3]\quad \begin{matrix}\mathcal{D}_{212}\\ \tilde{\mathcal{E}}^\star = \mathcal{E}\end{matrix}\quad \begin{matrix}\mathcal{D}_{213}\\ \tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu\end{matrix}}{(\lambda t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu)^\star = (\lambda t', \mathcal{E}, \mathcal{E}_\mu)}\,[\text{clos}^\star]\quad \dfrac{\begin{matrix}\mathcal{D}_{221}\\ \tilde c^\star = c\end{matrix}\quad \begin{matrix}\mathcal{D}_{222}\\ \tilde{\mathcal{S}}^\star = \mathcal{S}\end{matrix}}{(\tilde c :: \tilde{\mathcal{S}})^\star = c :: \mathcal{S}}\,[\text{stack}^\star_2]}{\langle \lambda t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, \tilde c :: \tilde{\mathcal{S}}\rangle^\star = \langle \lambda t', \mathcal{E}, \mathcal{E}_\mu, c :: \mathcal{S}\rangle}\,[\text{state}^\star]\ \Rightarrow\ \langle \lambda t', \mathcal{E}, \mathcal{E}_\mu, c :: \mathcal{S}\rangle \rightsquigarrow \langle t', (c; \mathcal{E}), \mathcal{E}_\mu, \mathcal{S}\rangle\ ^{[\text{k·abs}]}\ \wedge\ \dfrac{\dfrac{\begin{matrix}\mathcal{D}_{2111}\\ \downarrow_{n+1}^{(n+1 :: \mathcal{I}),\mathcal{I}_\mu}(t) = t'\end{matrix}\quad \dfrac{\begin{matrix}\mathcal{D}_{221}\\ \tilde c^\star = c\end{matrix}\quad \begin{matrix}\mathcal{D}_{212}\\ \tilde{\mathcal{E}}^\star = \mathcal{E}\end{matrix}}{(\tilde c; \tilde{\mathcal{E}})^\star = (c; \mathcal{E})}\,[\text{c·env}^\star_2]\quad \begin{matrix}\mathcal{D}_{213}\\ \tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu\end{matrix}}{(t, n+1, (n+1 :: \mathcal{I}), \mathcal{I}_\mu, (\tilde c; \tilde{\mathcal{E}}), \tilde{\mathcal{E}}_\mu)^\star = (t', (c; \mathcal{E}), \mathcal{E}_\mu)}\,[\text{clos}^\star]\quad \begin{matrix}\mathcal{D}_{222}\\ \tilde{\mathcal{S}}^\star = \mathcal{S}\end{matrix}}{\langle t, n+1, (n+1 :: \mathcal{I}), \mathcal{I}_\mu, (\tilde c; \tilde{\mathcal{E}}), \tilde{\mathcal{E}}_\mu, \tilde{\mathcal{S}}\rangle^\star = \langle t', (c; \mathcal{E}), \mathcal{E}_\mu, \mathcal{S}\rangle}\,[\text{state}^\star]\quad ^{[\text{soundness}]}}\,[\&3]$$

$$\dfrac{}{\langle \textbf{get-context}\,t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, \tilde{\mathcal{S}}\rangle \rightsquigarrow \langle t, n, \mathcal{I}, (\mathcal{I} :: \mathcal{I}_\mu), \tilde{\mathcal{E}}, (\tilde{\mathcal{S}}; \tilde{\mathcal{E}}_\mu), \tilde{\mathcal{S}}\rangle\ ^{[\text{i·catch}]}\ \wedge\ \dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_{2111}\\ \downarrow_n^{\mathcal{I},(\mathcal{I} :: \mathcal{I}_\mu)}(t) = t'\end{matrix}}{\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(\textbf{get-context}\,t) = \textbf{catch}\,t'}\,[\downarrow_4]\quad \begin{matrix}\mathcal{D}_{212}\\ \tilde{\mathcal{E}}^\star = \mathcal{E}\end{matrix}\quad \begin{matrix}\mathcal{D}_{213}\\ \tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu\end{matrix}}{(\textbf{get-context}\,t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu)^\star = (\textbf{catch}\,t', \mathcal{E}, \mathcal{E}_\mu)}\,[\text{clos}^\star]\quad \begin{matrix}\mathcal{D}_{22}\\ \tilde{\mathcal{S}}^\star = \mathcal{S}\end{matrix}}{\langle \textbf{get-context}\,t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, \tilde{\mathcal{S}}\rangle^\star = \langle \textbf{catch}\,t', \mathcal{E}, \mathcal{E}_\mu, \mathcal{S}\rangle}\,[\text{state}^\star]\ \Rightarrow\ \langle \textbf{catch}\,t', \mathcal{E}, \mathcal{E}_\mu, \mathcal{S}\rangle \rightsquigarrow \langle t', \mathcal{E}, (\mathcal{S}; \mathcal{E}_\mu), \mathcal{S}\rangle\ ^{[\text{k·catch}]}\ \wedge\ \dfrac{\dfrac{\begin{matrix}\mathcal{D}_{2111}\\ \downarrow_n^{\mathcal{I},(\mathcal{I} :: \mathcal{I}_\mu)}(t) = t'\end{matrix}\quad \begin{matrix}\mathcal{D}_{212}\\ \tilde{\mathcal{E}}^\star = \mathcal{E}\end{matrix}\quad \dfrac{\begin{matrix}\mathcal{D}_{22}\\ \tilde{\mathcal{S}}^\star = \mathcal{S}\end{matrix}\quad \begin{matrix}\mathcal{D}_{213}\\ \tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu\end{matrix}}{(\tilde{\mathcal{S}}; \tilde{\mathcal{E}}_\mu)^\star = (\mathcal{S}; \mathcal{E}_\mu)}\,[\text{k·env}^\star_2]}{(t, n, \mathcal{I}, (\mathcal{I} :: \mathcal{I}_\mu), \tilde{\mathcal{E}}, (\tilde{\mathcal{S}}; \tilde{\mathcal{E}}_\mu))^\star = (t', \mathcal{E}, (\mathcal{S}; \mathcal{E}_\mu))}\,[\text{clos}^\star]\quad \begin{matrix}\mathcal{D}_{22}\\ \tilde{\mathcal{S}}^\star = \mathcal{S}\end{matrix}}{\langle t, n, \mathcal{I}, (\mathcal{I} :: \mathcal{I}_\mu), \tilde{\mathcal{E}}, (\tilde{\mathcal{S}}; \tilde{\mathcal{E}}_\mu), \tilde{\mathcal{S}}\rangle^\star = \langle t', \mathcal{E}, (\mathcal{S}; \mathcal{E}_\mu), \mathcal{S}\rangle}\,[\text{state}^\star]\quad ^{[\text{soundness}]}}\,[\&4]$$

$$\dfrac{\begin{matrix}\mathcal{D}_{213}\\ \tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu\end{matrix}\ \wedge\ \begin{matrix}\mathcal{D}_{12}\\ \tilde{\mathcal{E}}_\mu(\alpha) = \tilde{\mathcal{S}}'\end{matrix}\ \Rightarrow\ \begin{matrix}\mathcal{D}_{32}\\ \tilde{\mathcal{S}}'^\star = \mathcal{S}'\end{matrix}\ \wedge\ \begin{matrix}\mathcal{D}_{4}\\ \mathcal{E}_\mu(\alpha) = \mathcal{S}'\end{matrix}\quad ^{[\text{fetch}^\mu\text{·sound}]}}{\dfrac{\begin{matrix}\mathcal{D}_{11}\\ \mathcal{I}_\mu(\alpha) = \mathcal{I}'\end{matrix}\quad \begin{matrix}\mathcal{D}_{12}\\ \tilde{\mathcal{E}}_\mu(\alpha) = \tilde{\mathcal{S}}'\end{matrix}}{\langle \textbf{set-context}\,\alpha\,t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, \tilde{\mathcal{S}}\rangle \rightsquigarrow \langle t, n, \mathcal{I}', \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, \tilde{\mathcal{S}}'\rangle}\,[\text{i·throw}]\ \wedge\ \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \mathcal{I}_\mu(\alpha) = \mathcal{I}'\end{matrix}\quad \begin{matrix}\mathcal{D}_{2112}\\ \downarrow_n^{\mathcal{I}',\mathcal{I}_\mu}(t) = t'\end{matrix}}{\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(\textbf{set-context}\,\alpha\,t) = \textbf{throw}\,\alpha\,t'}\,[\downarrow_5]\quad \begin{matrix}\mathcal{D}_{212}\\ \tilde{\mathcal{E}}^\star = \mathcal{E}\end{matrix}\quad \begin{matrix}\mathcal{D}_{213}\\ \tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu\end{matrix}}{(\textbf{set-context}\,\alpha\,t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu)^\star = (\textbf{throw}\,\alpha\,t', \mathcal{E}, \mathcal{E}_\mu)}\,[\text{clos}^\star]\quad \begin{matrix}\vdots\\ \tilde{\mathcal{S}}^\star = \mathcal{S}\end{matrix}}{\langle \textbf{set-context}\,\alpha\,t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, \tilde{\mathcal{S}}\rangle^\star = \langle \textbf{throw}\,\alpha\,t', \mathcal{E}, \mathcal{E}_\mu, \mathcal{S}\rangle}\,[\text{state}^\star]\ \Rightarrow\ \dfrac{\begin{matrix}\mathcal{D}_{4}\\ \mathcal{E}_\mu(\alpha) = \mathcal{S}'\end{matrix}}{\langle \textbf{throw}\,\alpha\,t', \mathcal{E}, \mathcal{E}_\mu, \mathcal{S}\rangle \rightsquigarrow \langle t', \mathcal{E}, \mathcal{E}_\mu, \mathcal{S}'\rangle}\,[\text{k·throw}]\ \wedge\ \dfrac{\dfrac{\begin{matrix}\mathcal{D}_{2112}\\ \downarrow_n^{\mathcal{I}',\mathcal{I}_\mu}(t) = t'\end{matrix}\quad \begin{matrix}\mathcal{D}_{212}\\ \tilde{\mathcal{E}}^\star = \mathcal{E}\end{matrix}\quad \begin{matrix}\mathcal{D}_{213}\\ \tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu\end{matrix}}{(t, n, \mathcal{I}', \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu)^\star = (t', \mathcal{E}, \mathcal{E}_\mu)}\,[\text{clos}^\star]\quad \begin{matrix}\mathcal{D}_{32}\\ \tilde{\mathcal{S}}'^\star = \mathcal{S}'\end{matrix}}{\langle t, n, \mathcal{I}', \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, \tilde{\mathcal{S}}'\rangle^\star = \langle t', \mathcal{E}, \mathcal{E}_\mu, \mathcal{S}'\rangle}\,[\text{state}^\star]\quad ^{[\text{soundness}]}}\,[\&5]$$

%mode $+\mathcal{D}_1\ \wedge\ +\mathcal{D}_2\ \Rightarrow\ -\mathcal{D}_3\ \wedge\ -\mathcal{D}_4\ ^{[\text{soundness}]}$
%worlds $()\ \ \mathcal{D}_1\ \wedge\ \mathcal{D}_2\ \Rightarrow\ \mathcal{D}_3\ \wedge\ \mathcal{D}_4\ ^{[\text{soundness}]}$
%total $(\mathcal{D}_1)\ \ \mathcal{D}_1\ \wedge\ \mathcal{D}_2\ \Rightarrow\ \mathcal{D}_3\ \wedge\ \mathcal{D}_4\ ^{[\text{soundness}]}$ □