1 First simulation (completeness)

1.1 Syntax

1.1.1 Index, indices and tables

%datatype *index*
%name *index* n α

$$\begin{array}{lcl} n & ::= & 0 \\ & | & n+1 \end{array}$$

%datatype *vector*
%name *vector* $\mathcal{I}$

$$\begin{array}{lcl} \mathcal{I} & ::= & [\,] \\ & | & n :: \mathcal{I} \end{array}$$

%datatype *table*
%name *table* $\mathcal{I}_\mu$

$$\begin{array}{lcl} \mathcal{I}_\mu & ::= & [\,] \\ & | & \mathcal{I} :: \mathcal{I}_\mu \end{array}$$

1.1.2 Term

%datatype *term*
%name *term* t

$$\begin{array}{lcl} t & ::= & n \\ & | & t_1\, t_2 \\ & | & \lambda t \\ & | & \mathbf{catch}\, t \\ & | & \mathbf{throw}\, \alpha\, t \end{array}$$

Remark. Syntax of safe λ_{ct}-terms:

$$\begin{array}{lcl} t & ::= & n \\ & | & t_1\, t_2 \\ & | & \lambda t \\ & | & \mathbf{get\text{-}context}\, t \\ & | & \mathbf{set\text{-}context}\, \alpha\, t \end{array}$$

1.2 Subtraction

%judgment $n_1 \dot{-} n_2 = n_3$

$$n_1 \dot{-} 0 = n_1 \ ^{[\mathrm{minus}_1]}$$

$$(n_1+1) \dot{-} (n_2+1) = n_3 \ ^{[\mathrm{minus}_2]} \quad \textit{when} \quad n_1 \dot{-} n_2 = n_3$$

%mode $+n_1 \dot{-} +n_2 = -n_3$

%worlds $()$ $n_1 \dot{-} n_2 = n_3$

%terminates (n_1) $n_1 \dot{-} n_2 = n_3$

%unique $+n_1 \dot{-} +n_2 = -1n_3$

1.2.1 Fetch (indices)

%judgment $\mathcal{I}(n_1) = n_2$

$$(n :: \mathcal{I})(0) = n \;{}^{[\mathrm{fetch}_1^{\mathcal{I}}]}$$
$$(n :: \mathcal{I})(n_1 + 1) = n_2 \;{}^{[\mathrm{fetch}_2^{\mathcal{I}}]} \quad \textit{when} \quad \mathcal{I}(n_1) = n_2$$

%mode $+\mathcal{I}(+n_1) = -n_2$
%worlds $()\quad \mathcal{I}(n_1) = n_2$
%terminates $n_1 \quad \mathcal{I}(n_1) = n_2$
%unique $+\mathcal{I}(+n_1) = -1n_2$

1.2.2 Fetch (table)

%judgment $\mathcal{I}_\mu(n) = \mathcal{I}$

$$(\mathcal{I} :: \mathcal{I}_\mu)(0) = \mathcal{I} \;{}^{\left[\mathrm{fetch}_1^{\mathcal{I}_\mu}\right]}$$
$$(\mathcal{I}' :: \mathcal{I}_\mu)(\alpha + 1) = \mathcal{I} \;{}^{\left[\mathrm{fetch}_2^{\mathcal{I}_\mu}\right]} \quad \textit{when} \quad \mathcal{I}_\mu(\alpha) = \mathcal{I}$$

%mode $+\mathcal{I}_\mu(+\alpha) = -\mathcal{I}$
%worlds $()\quad \mathcal{I}_\mu(\alpha) = \mathcal{I}$
%terminates $\alpha \quad \mathcal{I}_\mu(\alpha) = \mathcal{I}$
%unique $+\mathcal{I}_\mu(+\alpha) = -1\mathcal{I}$

1.2.3 Compute

%judgment $n_1 \dot{-} \mathcal{I}(n_2) = n_3$

$$n \dot{-} \mathcal{I}(l) = g \;{}^{[\mathrm{compute}_1]} \quad \textit{when} \quad \mathcal{I}(l) = k, \quad n \dot{-} k = g$$

%mode $+n_1 \dot{-} +\mathcal{I}(+n_2) = -n_3$
%worlds $()\quad n_1 \dot{-} \mathcal{I}(n_2) = n_3$
%terminates $\{\}\quad n_1 \dot{-} \mathcal{I}(n_2) = n_3$
%unique $+n_1 \dot{-} +\mathcal{I}(+n_2) = -1n_3$

1.3 From local indices to global indices

%judgment $\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t_1) = t_2$

$$\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(l) = g \;{}^{[\downarrow_1]} \quad \textit{when} \quad n \dot{-} \mathcal{I}(l) = g$$
$$\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t\,u) = t'\,u' \;{}^{[\downarrow_2]} \quad \textit{when} \quad \downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t) = t', \quad \downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(u) = u'$$
$$\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(\lambda t) = \lambda t' \;{}^{[\downarrow_3]} \quad \textit{when} \quad \downarrow_{n+1}^{(n+1 :: \mathcal{I}),\mathcal{I}_\mu}(t) = t'$$
$$\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(\mathbf{get\text{-}context}\; t) = \mathbf{catch}\; t' \;{}^{[\downarrow_4]} \quad \textit{when} \quad \downarrow_n^{\mathcal{I},(\mathcal{I} :: \mathcal{I}_\mu)}(t) = t'$$
$$\downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(\mathbf{set\text{-}context}\; \alpha\; t) = \mathbf{throw}\; \alpha\; t' \;{}^{[\downarrow_5]} \quad \textit{when} \quad \mathcal{I}_\mu(\alpha) = \mathcal{I}', \quad \downarrow_n^{\mathcal{I}',\mathcal{I}_\mu}(t) = t'$$

%mode $\downarrow_{+n}^{+\mathcal{I},+\mathcal{I}_\mu}(+t) = -t'$
%worlds $()\quad \downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t) = t'$
%terminates $t \quad \downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t) = t'$
%unique $\downarrow_{+n}^{+\mathcal{I},+\mathcal{I}_\mu}(+t) = -1t'$

1.3.1 Closure, environment and stack

%datatype *clos* **%name** *clos* c
%datatype *c-env* **%name** *c-env* $\mathcal{E}$
%datatype *k-env* **%name** *k-env* $\mathcal{E}_\mu$
%datatype *stack* **%name** *stack* $\mathcal{S}$

$$
\begin{array}{lcl}
c & ::= & (t, \mathcal{E}, \mathcal{E}_\mu) \\
\\
\mathcal{E} & ::= & () \\
 & | & (c; \mathcal{E}) \\
\\
\mathcal{E}_\mu & ::= & () \\
 & | & (\mathcal{S}; \mathcal{E}_\mu) \\
\\
\mathcal{S} & ::= & [] \\
 & | & c :: \mathcal{S}
\end{array}
$$

%datatype *state*
%name *state* σ

$$\sigma \quad ::= \quad \langle t, \mathcal{E}, \mathcal{E}_\mu, \mathcal{S} \rangle$$

1.4 Judgments

1.4.1 Fetch a closure

%judgment $\mathcal{E}(n) = c$

$$(c; \mathcal{E})(0) = c \ ^{[\text{fetch}_1]}$$
$$(c'; \mathcal{E})(n+1) = c \ ^{[\text{fetch}_2]} \quad \textit{when} \quad \mathcal{E}(n) = c$$

%mode $+\mathcal{E}(+n) = -c$
%worlds $()$ $\mathcal{E}(n) = c$
%terminates $\mathcal{E}$ $\mathcal{E}(n) = c$
%unique $+\mathcal{E}(+n) = -1c$

1.4.2 Fetch a stack

%judgment $\mathcal{E}_\mu(n) = \mathcal{S}$

$$(\mathcal{S}; \mathcal{E}_\mu)(0) = \mathcal{S} \ ^{[\text{fetch}_1^\mu]}$$
$$(\mathcal{S}'; \mathcal{E}_\mu)(n+1) = \mathcal{S} \ ^{[\text{fetch}_2^\mu]} \quad \textit{when} \quad \mathcal{E}_\mu(n) = \mathcal{S}$$

%mode $+\mathcal{E}_\mu(+n) = -\mathcal{S}$
%worlds $()$ $\mathcal{E}_\mu(n) = \mathcal{S}$
%terminates $\mathcal{E}_\mu$ $\mathcal{E}_\mu(n) = \mathcal{S}$
%unique $+\mathcal{E}_\mu(+n) = -1\mathcal{S}$

1.4.3 Evaluation rules

%judgment $\sigma_1 \rightsquigarrow \sigma_2$

$$\langle k, \mathcal{E}, \mathcal{E}_\mu, \mathcal{S} \rangle \rightsquigarrow \langle t, \mathcal{E}', \mathcal{E}'_\mu, \mathcal{S} \rangle \ ^{[\text{k}\cdot\text{var}]} \quad \textit{when} \quad \mathcal{E}(k) = (t, \mathcal{E}', \mathcal{E}'_\mu)$$
$$\langle (t\, u), \mathcal{E}, \mathcal{E}_\mu, \mathcal{S} \rangle \rightsquigarrow \langle t, \mathcal{E}, \mathcal{E}_\mu, (u, \mathcal{E}, \mathcal{E}_\mu) :: \mathcal{S} \rangle \ ^{[\text{k}\cdot\text{app}]}$$

$$\langle \lambda t, \mathcal{E}, \mathcal{E}_\mu, c :: \mathcal{S}\rangle \rightsquigarrow \langle t, (c; \mathcal{E}), \mathcal{E}_\mu, \mathcal{S}\rangle^{\text{[k·abs]}}$$
$$\langle \mathbf{catch}\, t, \mathcal{E}, \mathcal{E}_\mu, \mathcal{S}\rangle \rightsquigarrow \langle t, \mathcal{E}, (\mathcal{S}; \mathcal{E}_\mu), \mathcal{S}\rangle^{\text{[k·catch]}}$$
$$\langle \mathbf{throw}\, \alpha\, t, \mathcal{E}, \mathcal{E}_\mu, \mathcal{S}\rangle \rightsquigarrow \langle t, \mathcal{E}, \mathcal{E}_\mu, \mathcal{S}'\rangle^{\text{[k·throw]}} \quad \textit{when} \quad \mathcal{E}_\mu(\alpha) = \mathcal{S}'$$

%mode $+\sigma_1 \rightsquigarrow -\sigma_2$
%worlds $()$ $\sigma_1 \rightsquigarrow \sigma_2$
%unique $+\sigma_1 \rightsquigarrow -1\sigma_2$

1.5 Abstract machine for safe $\boldsymbol{\lambda}_{\mathbf{ct}}$-terms

1.5.1 Syntax

%datatype *clos*
%datatype *c-env*
%datatype *k-env*
%datatype *stack*
%name *clos* $\tilde{c}$
%name *c-env* $\tilde{\mathcal{E}}$
%name *k-env* $\tilde{\mathcal{E}}_\mu$
%name *stack* $\tilde{\mathcal{S}}$

$$\tilde{c} ::= \big(t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu\big)$$

$$\tilde{\mathcal{E}} ::= () \mid (\tilde{c}; \tilde{\mathcal{E}})$$

$$\tilde{\mathcal{E}}_\mu ::= () \mid \big(\tilde{\mathcal{S}}; \tilde{\mathcal{E}}_\mu\big)$$

$$\tilde{\mathcal{S}} ::= [] \mid \tilde{c} :: \tilde{\mathcal{S}}$$

%datatype *state*
%name *state* $\tilde{\sigma}$

$$\tilde{\sigma} ::= \langle t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, \tilde{\mathcal{S}}\rangle$$

1.5.2 Fetch a closure

%judgment $\tilde{\mathcal{E}}(n) = \tilde{c}$

$$(\tilde{c}; \tilde{\mathcal{E}})(0) = \tilde{c}^{\text{[i·fetch}_1]}$$
$$(\tilde{c}'; \tilde{\mathcal{E}})(n+1) = \tilde{c}^{\text{[i·fetch}_2]} \quad \textit{when} \quad \tilde{\mathcal{E}}(n) = \tilde{c}$$

%mode $+\tilde{\mathcal{E}}(+n) = -\tilde{c}$
%worlds $()$ $\tilde{\mathcal{E}}(n) = \tilde{c}$
%terminates $\tilde{\mathcal{E}}$ $\tilde{\mathcal{E}}(n) = \tilde{c}$
%unique $+\tilde{\mathcal{E}}(+n) = -1\tilde{c}$

1.5.3 Fetch a stack

%judgment $\tilde{\mathcal{E}}_\mu(n) = \tilde{\mathcal{S}}$

$$\big(\tilde{\mathcal{S}}; \tilde{\mathcal{E}}_\mu\big)(0) = \tilde{\mathcal{S}}^{\text{[i·fetch}_1^\mu]}$$

$$\big(\tilde{\mathcal{S}}';\tilde{\mathcal{E}}_\mu\big)(n+1)=\tilde{\mathcal{S}}^{\,[\text{i·fetch}_2^\mu]} \quad \textit{when} \quad \tilde{\mathcal{E}}_\mu(n)=\tilde{\mathcal{S}}$$

%mode $+\tilde{\mathcal{E}}_\mu(+n)=-\tilde{\mathcal{S}}$
%worlds $()$ $\tilde{\mathcal{E}}_\mu(n)=\tilde{\mathcal{S}}$
%terminates $\tilde{\mathcal{E}}_\mu$ $\tilde{\mathcal{E}}_\mu(n)=\tilde{\mathcal{S}}$
%unique $+\tilde{\mathcal{E}}_\mu(+n)=-1\tilde{\mathcal{S}}$

2 Evaluation rules

%judgment $\tilde{\sigma}_1 \rightsquigarrow \tilde{\sigma}_2$

$$\langle l,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu,\tilde{\mathcal{S}}\rangle \rightsquigarrow \langle t,n',\mathcal{I}',\mathcal{I}'_\mu,\tilde{\mathcal{E}}',\tilde{\mathcal{E}}'_\mu,\tilde{\mathcal{S}}\rangle^{\,[\text{i·var}]}$$
$$\textit{when} \quad n \dot{-} \mathcal{I}(l)=g, \quad \tilde{\mathcal{E}}(g)=\big(t,n',\mathcal{I}',\mathcal{I}'_\mu,\tilde{\mathcal{E}}',\tilde{\mathcal{E}}'_\mu\big)$$

$$\langle(t\,u),n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu,\tilde{\mathcal{S}}\rangle \rightsquigarrow \langle t,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu,\big(u,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu\big)::\tilde{\mathcal{S}}\rangle^{\,[\text{i·app}]}$$

$$\langle \lambda t,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu,\tilde{c}::\tilde{\mathcal{S}}\rangle \rightsquigarrow \langle t,n+1,(n+1::\mathcal{I}),\mathcal{I}_\mu,(\tilde{c};\tilde{\mathcal{E}}),\tilde{\mathcal{E}}_\mu,\tilde{\mathcal{S}}\rangle^{\,[\text{i·abs}]}$$

$$\langle \textbf{get-context}\,t,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu,\tilde{\mathcal{S}}\rangle \rightsquigarrow \langle t,n,\mathcal{I},(\mathcal{I}::\mathcal{I}_\mu),\tilde{\mathcal{E}},\big(\tilde{\mathcal{S}};\tilde{\mathcal{E}}_\mu\big),\tilde{\mathcal{S}}\rangle^{\,[\text{i·catch}]}$$

$$\langle \textbf{set-context}\,\alpha\,t,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu,\tilde{\mathcal{S}}\rangle \rightsquigarrow \langle t,n,\mathcal{I}',\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu,\tilde{\mathcal{S}}'\rangle^{\,[\text{i·throw}]}$$
$$\textit{when} \quad \mathcal{I}_\mu(\alpha)=\mathcal{I}', \quad \tilde{\mathcal{E}}_\mu(\alpha)=\tilde{\mathcal{S}}'$$

%mode $+\sigma_1 \rightsquigarrow -\sigma_2$
%worlds $()$ $\sigma_1 \rightsquigarrow \sigma_2$
%unique $+\sigma_1 \rightsquigarrow -1\sigma_2$

3 Translation

%judgment $\tilde{c}^\star=c$
%judgment $\tilde{\mathcal{S}}^\star=\mathcal{S}$
%judgment $\tilde{\mathcal{E}}^\star=\mathcal{E}$
%judgment $\tilde{\mathcal{E}}_\mu{}^\star=\mathcal{E}_\mu$

$$\big(t,n,\mathcal{I},\mathcal{I}_\mu,\tilde{\mathcal{E}},\tilde{\mathcal{E}}_\mu\big)^\star=(u,\mathcal{E},\mathcal{E}_\mu)^{\,[\text{clos}^\star]} \quad \textit{when} \quad \downarrow_n^{\mathcal{I},\mathcal{I}_\mu}(t)=u, \quad \tilde{\mathcal{E}}^\star=\mathcal{E}, \quad \tilde{\mathcal{E}}_\mu{}^\star=\mathcal{E}_\mu$$

$$[\,]^\star=[\,]^{\,[\text{stack}_1^\star]}$$
$$\big(\tilde{c}::\tilde{\mathcal{S}}\big)^\star=c::\mathcal{S}^{\,[\text{stack}_2^\star]} \quad \textit{when} \quad \tilde{c}^\star=c, \quad \tilde{\mathcal{S}}^\star=\mathcal{S}$$

$$()^\star=()^{\,[\text{c·env}_1^\star]}$$
$$(\tilde{c};\tilde{\mathcal{E}})^\star=(c;\mathcal{E})^{\,[\text{c·env}_2^\star]} \quad \textit{when} \quad \tilde{c}^\star=c, \quad \tilde{\mathcal{E}}^\star=\mathcal{E}$$

$$()^\star=()^{\,[\text{k·env}_1^\star]}$$
$$\big(\tilde{\mathcal{S}};\tilde{\mathcal{E}}_\mu\big)^\star=(\mathcal{S};\mathcal{E}_\mu)^{\,[\text{k·env}_2^\star]} \quad \textit{when} \quad \tilde{\mathcal{S}}^\star=\mathcal{S}, \quad \tilde{\mathcal{E}}_\mu{}^\star=\mathcal{E}_\mu$$

%mode
$+\,\tilde{c}^\star=-c$
$+\,\tilde{\mathcal{S}}^\star=-\mathcal{S}$
$+\,\tilde{\mathcal{E}}^\star=-\mathcal{E}$
$+\,\tilde{\mathcal{E}}_\mu{}^\star=-\mathcal{E}_\mu$

%worlds ()
$\tilde{c}^\star = c$
$\tilde{\mathcal{S}}^\star = \mathcal{S}$
$\tilde{\mathcal{E}}^\star = \mathcal{E}$
$\tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu$

%terminates $\big(\tilde{c}\ \tilde{\mathcal{S}}\ \tilde{\mathcal{E}}\ \tilde{\mathcal{E}}_\mu\big)$
$\tilde{c}^\star = c$
$\tilde{\mathcal{S}}^\star = \mathcal{S}$
$\tilde{\mathcal{E}}^\star = \mathcal{E}$
$\tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu$

%unique
$+\,\tilde{c}^\star = -1c$
$+\,\tilde{\mathcal{S}}^\star = -1\mathcal{S}$
$+\,\tilde{\mathcal{E}}^\star = -1\mathcal{E}$
$+\,\tilde{\mathcal{E}}_\mu{}^\star = -1\mathcal{E}_\mu$

%judgment $\tilde{\sigma}^\star = \sigma$
$\langle t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu, \tilde{\mathcal{S}}\rangle^\star = \langle u, \mathcal{E}, \mathcal{E}_\mu, \mathcal{S}\rangle$ [state⋆] *when* $\big(t, n, \mathcal{I}, \mathcal{I}_\mu, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_\mu\big)^\star = (u, \mathcal{E}, \mathcal{E}_\mu)$, $\tilde{\mathcal{S}}^\star = \mathcal{S}$

%mode $+\tilde{\sigma}^\star = -\sigma$
%worlds () $\tilde{\sigma}^\star = \sigma$
%unique $+\tilde{\sigma}^\star = -1\sigma$

4 Completeness

%lemma $\tilde{\mathcal{E}}^\star = \mathcal{E} \quad \wedge \quad \mathcal{E}(n) = c \quad \Rightarrow \quad \tilde{\mathcal{E}}(n) = \tilde{c} \quad \wedge \quad \tilde{c}^\star = c$ *for some* $\tilde{c}$ [fetch·complete]

%lemma $\tilde{\mathcal{E}}_\mu{}^\star = \mathcal{E}_\mu \quad \wedge \quad \mathcal{E}_\mu(\alpha) = \mathcal{S} \quad \Rightarrow \quad \tilde{\mathcal{E}}_\mu(\alpha) = \tilde{\mathcal{S}} \quad \wedge \quad \tilde{\mathcal{S}}^\star = \mathcal{S}$ *for some* $\tilde{\mathcal{S}}$ [fetch$^\mu$·complete]

%theorem $\sigma_1 \rightsquigarrow \sigma_2 \quad \wedge \quad \tilde{\sigma}_1{}^\star = \sigma_1 \quad \Rightarrow \quad \tilde{\sigma}_1 \rightsquigarrow \tilde{\sigma}_2 \quad \wedge \quad \tilde{\sigma}_2{}^\star = \sigma_2$ *for some* $\tilde{\sigma}_2$ [completeness]