Monadic reflection in Lax Logic

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Introduction

- Revisit "*Representing monads*" [Filinski, 1994] from a logical standpoint.
- **Goal:** understand the logical meaning of **shift**/**reset** in the restricted framework of a major application, i.e. implementing monadic reflection.
- Through the formulas-as-types interpretation, a monad $\Diamond\,\sqcup$ corresponds to the modality from Lax logic [Curry, 1952]:

$$\vdash unit : \varphi \Rightarrow \Diamond\varphi \qquad\qquad \vdash bind : (\varphi \Rightarrow \Diamond\psi) \Rightarrow \Diamond\varphi \Rightarrow \Diamond\psi$$

- Monadic reflection is given by these logical rules:

$$\frac{\Gamma \vdash\ t : \varphi}{\Gamma \vdash\ [t] : \Diamond\varphi}\,[\text{reify}] \qquad \frac{\Gamma \vdash\ t : \Diamond\varphi}{\Gamma \vdash\ \mu(t) : \varphi}\,[\text{reflect}]$$

- In this talk, we consider only *provability*.

Example: the exception monad

$$\begin{aligned}\varepsilon &: type.\\ \Diamond\varphi &= \varphi \vee \varepsilon.\end{aligned}$$

$$\begin{aligned}unit &= \lambda a\colon \varphi.\mathbf{inl}\, a.\\ bind\, f &= \lambda t.\mathbf{case}\, t\, \mathbf{of}\ (\mathbf{inl}\, a) \mapsto f\ \ a \mid (\mathbf{inr}\, b) \mapsto \mathbf{inr}\, b.\end{aligned}$$

Defining **raise** and **handle** in direct style:

$$\mathbf{raise} = \mu(\mathbf{inr}\, t).$$

$$t\,\mathbf{handle}\, e \mapsto h = \mathbf{case}\ [t]\ \mathbf{of}\ (\mathbf{inl}\, a) \mapsto a \mid (\mathbf{inr}\, e) \mapsto h.$$

Moggi's monadic translation (CBV)

Translation of types (σ atomic):

- $\sigma^{\Diamond} \equiv \sigma$
- $(\varphi \Rightarrow \psi)^{\Diamond} \equiv \varphi^{\Diamond} \Rightarrow \Diamond \psi^{\Diamond}$
- $(\Diamond \psi)^{\Diamond} \equiv \Diamond \varphi^{\Diamond}$

Translation of terms:

- $x^{\Diamond} \equiv \mathit{unit}\ x$
- $(\lambda x.t)^{\Diamond} \equiv \mathit{unit}\ \lambda x.t^{\Diamond}$
- $(t_1\ t_2)^{\Diamond} \equiv \mathit{bind}\big(\lambda f.\big(\mathit{bind}\ f\ t_2^{\Diamond}\big)\big)\ t_1^{\Diamond}$
- $\mu(t)^{\Diamond} \equiv \mathit{bind}\ \mathit{id}\ t^{\Diamond}$
- $[t]^{\Diamond} \equiv \mathit{unit}\ t^{\Diamond}$

Lemma. *If $\Gamma \vdash t{:}\,\varphi$ is derivable then $\Gamma^{\Diamond} \vdash t^{\Diamond}{:}\,\Diamond\varphi^{\Diamond}$ is derivable.*

Proof.

$$\frac{\Gamma^{\Diamond} \vdash t^{\Diamond} : \Diamond\varphi^{\Diamond}}{\Gamma^{\Diamond} \vdash \mathit{unit}\ t^{\Diamond} : \Diamond\Diamond\varphi^{\Diamond}}\,[\text{unit}] \qquad \frac{\Gamma^{\Diamond} \vdash t^{\Diamond} : \Diamond\Diamond\varphi^{\Diamond}}{\Gamma^{\Diamond} \vdash \mathit{bind}\ \mathit{id}\ t^{\Diamond} : \Diamond\varphi^{\Diamond}}\,[\text{join}]$$

Filinski's CPS-translation (CBV)

Define $\nabla\varphi = (\varphi \Rightarrow \Diamond o) \Rightarrow \Diamond o$ where o is some universal answer type (not sound).

Translation of types (σ atomic):

- $\sigma^\nabla \equiv \sigma$
- $(\varphi \Rightarrow \psi)^\nabla \equiv \varphi^\nabla \Rightarrow \nabla\psi^\nabla$
- $(\Diamond\psi)^\nabla \equiv \Diamond\varphi^\nabla$

Translation of terms:

- $x^\nabla \equiv \lambda k.k\ x$
- $(\lambda x.t)^\nabla \equiv \lambda k.k\ (\lambda x.t^\nabla)$
- $(t_1\ t_2)^\nabla \equiv \lambda k.t_1^\nabla\,(\lambda f.t_2^\nabla\,(\lambda a.f\ a\ k))$
- $\mu(t)^\nabla \equiv \lambda k.t^\nabla\,(bind\,k)$
- $[t]^\nabla \equiv \lambda k.k\,(t^\nabla\ unit)$

Delimited control

Reflect/reify are *definable in direct style* from **shift**/**reset** [Filinski, 1994]

$$\begin{array}{rcl} [t] & = & \textbf{reset}\ (\mathit{unit}\ t). \\ \mu(t) & = & \textbf{shift}\ (\lambda k.\mathit{bind}\ k\ t). \end{array}$$

That is, these equations are valid:

- $(\textbf{reset}\ (\mathit{unit}\ t))^\nabla = [t]^\nabla$
- $(\textbf{shift}\ (\lambda k.\mathit{bind}\ k\ t))^\nabla = \mu(t)^\nabla$

where:

- $\textbf{reset}^\nabla = \lambda m.\lambda c.(c\,(m\ \mathit{id}))$
- $\textbf{shift}^\nabla = \lambda h.\lambda c.(h\,(\lambda v.\lambda c'.c'\,(c\,v))\ \mathit{id})$

Answer type polymorphism

Footnote from "*Representing monads*":

> "Alternatively, with a little more care, we can take $\nabla\varphi = \forall\alpha(\varphi \to \Diamond\alpha) \to \Diamond\alpha$; it is straightforward to check that both the term translation and the operations defined in the following can in fact be typed according to this schema."

So, let us do it *carefully*:

- Formalization in Twelf (work in progress)
- Experimenting with TeXmacs as a front end
  (the Twelf source is generated from the slides)

Plan of the rest of the talk

- Formalize System F in Twelf
- Check that the operations are well-typed in direct style
- Check that the CPS-translations of the operations are well-typed
- Interpret the logical type of **shift** for the usual monads:
  - continuation monad
  - state monad
  - exception monad

System $\boldsymbol{F}$ (HOAS)

Types

$$
\begin{array}{lcl}
\mathit{type} & : & \mathbf{type}. \\
\sqcup \Rightarrow \sqcup & : & \mathit{type} \longrightarrow \mathit{type} \longrightarrow \mathit{type}. \\
\sqcup \wedge \sqcup & : & \mathit{type} \longrightarrow \mathit{type} \longrightarrow \mathit{type}. \\
\sqcup \vee \sqcup & : & \mathit{type} \longrightarrow \mathit{type} \longrightarrow \mathit{type}.
\end{array}
$$

$$
\begin{array}{lcl}
\forall \sqcup . \sqcup & : & (\mathit{type} \longrightarrow \mathit{type}) \longrightarrow \mathit{type}.
\end{array}
$$

$\textbf{\%binding} \quad 1 \mapsto 2 \quad \textbf{in} \quad \forall \sqcup . \sqcup$

$$
\mathbf{void} \quad = \quad \forall \beta . \beta .
$$

Terms

$term \quad : \quad \mathbf{type}.$

Abstraction

$\lambda \sqcup : \sqcup . \sqcup \quad : \quad type \;\rightarrow\; (term \;\rightarrow\; term) \;\rightarrow\; term.$

$\textbf{\%binding} \quad 1 \mapsto 3 \quad \textbf{in} \quad \lambda \sqcup : \sqcup . \sqcup$

Application

$\sqcup \;\; \sqcup \quad : \quad term \;\rightarrow\; term \;\rightarrow\; term.$

Polymorphic abstraction

$\Lambda \sqcup . \sqcup \quad : \quad (type \;\rightarrow\; term) \;\rightarrow\; term.$

$\textbf{\%binding} \quad 1 \mapsto 2 \quad \textbf{in} \quad \Lambda \sqcup . \sqcup$

Instantiation

$$\sqcup\{\sqcup\} \quad : \quad \mathit{term} \;\rightarrow\; \mathit{type} \;\rightarrow\; \mathit{term}.$$

Derived syntax for **let**

$$\textbf{let } \sqcup : \sqcup = \sqcup \textbf{ in } \sqcup \quad = \quad [\tau]\,[u]\,[t]\,(\lambda x{:}\tau.t[x])\ u.$$

$$\textbf{\%binding} \quad 1 \mapsto 4 \quad \textbf{in} \quad \textbf{let } \sqcup : \sqcup = \sqcup \textbf{ in } \sqcup$$

Pairing

$$\langle \sqcup , \sqcup \rangle \quad : \quad \mathit{term} \;\rightarrow\; \mathit{term} \;\rightarrow\; \mathit{term}.$$

Pattern matching

$$\textbf{let } \langle \sqcup , \sqcup \rangle = \sqcup \textbf{ in } \sqcup \quad : \quad \mathit{term} \;\rightarrow\; (\mathit{term} \;\rightarrow\; \mathit{term} \;\rightarrow\; \mathit{term}) \;\rightarrow\; \mathit{term}.$$

$$\textbf{\%binding} \quad 2 \mapsto 4 \quad \textbf{in} \quad \textbf{let } \langle \sqcup , \sqcup \rangle = \sqcup \textbf{ in } \sqcup$$
$$\textbf{\%binding} \quad 1 \mapsto 4 \quad \textbf{in} \quad \textbf{let } \langle \sqcup , \sqcup \rangle = \sqcup \textbf{ in } \sqcup$$

Injections

$$\mathbf{inl}\ {\sqcup} \quad : \quad \mathit{term} \ \rightarrow \ \mathit{term}.$$
$$\mathbf{inr}\ {\sqcup} \quad : \quad \mathit{term} \ \rightarrow \ \mathit{term}.$$

Pattern matching

$$\mathbf{case}\ {\sqcup}\ \mathbf{of}\ (\mathbf{inl}\ {\sqcup}) \mapsto {\sqcup} \mid (\mathbf{inr}\ {\sqcup}) \mapsto {\sqcup} \quad : \quad \mathit{term} \ \rightarrow \ (\mathit{term} \ \rightarrow \ \mathit{term}) \ \rightarrow \ (\mathit{term} \ \rightarrow \ \mathit{term}) \ \rightarrow \ \mathit{term}.$$

$$\%\mathbf{binding} \quad 4 \mapsto 5 \quad \mathbf{in} \quad \mathbf{case}\ {\sqcup}\ \mathbf{of}\ (\mathbf{inl}\ {\sqcup}) \mapsto {\sqcup} \mid (\mathbf{inr}\ {\sqcup}) \mapsto {\sqcup}$$
$$\%\mathbf{binding} \quad 2 \mapsto 3 \quad \mathbf{in} \quad \mathbf{case}\ {\sqcup}\ \mathbf{of}\ (\mathbf{inl}\ {\sqcup}) \mapsto {\sqcup} \mid (\mathbf{inr}\ {\sqcup}) \mapsto {\sqcup}$$

Monadic constants

$$\mathit{unit} \quad : \quad \mathit{term}.$$
$$\mathit{bind} \quad : \quad \mathit{term}.$$

Delimited control operators

$$\mathbf{reset} \quad : \quad \mathit{term}.$$
$$\mathbf{shift} \quad : \quad \mathit{term}.$$

Typing judgment

$$\vdash \sqcup : \sqcup \quad : \quad \mathit{term} \;\longrightarrow\; \mathit{type} \;\longrightarrow\; \mathbf{type}.$$

$$\frac{\{x\} \;\vdash x : \varphi \;\longrightarrow\; \vdash t[x] : \psi}{\vdash \lambda x{:}\,\varphi.t[x] : \varphi \Rightarrow \psi}\ [\&\mathrm{lam}]$$

$$\frac{\vdash t_1 : \varphi \Rightarrow \psi \qquad \vdash t_2 : \varphi}{\vdash t_1\ t_2 : \psi}\ [\&\mathrm{app}]$$

$$\frac{\{\alpha\} \;\vdash t[\alpha] : \psi[\alpha]}{\vdash \Lambda\alpha.t[\alpha] : \forall\alpha.\psi[\alpha]}\ [\&\mathrm{abs}]$$

$$\frac{\vdash t : \forall\alpha.\psi[\alpha]}{\vdash t\{\varphi\} : \psi[\varphi]}\ [\&\mathrm{inst}]$$

Typing judgment

$$\frac{\vdash t_1 : \varphi \quad \vdash t_2 : \psi}{\vdash \langle t_1, t_2 \rangle : \varphi \wedge \psi}\ [\&\mathrm{pair}]$$

$$\frac{\{x\} \vdash x : \varphi \;\rightarrow\; (\{y\} \vdash y : \psi \;\rightarrow\; \vdash u[x][y] : \tau) \quad \vdash t : \varphi \wedge \psi}{\vdash \mathbf{let}\ \langle x, y \rangle = t\ \mathbf{in}\ u[x][y] : \tau}\ [\&\mathrm{match}]$$

$$\frac{\vdash t : \psi}{\vdash \mathbf{inr}\, t : \varphi \vee \psi}\ [\&\mathrm{inr}]$$

$$\frac{\vdash t : \varphi}{\vdash \mathbf{inl}\, t : \varphi \vee \psi}\ [\&\mathrm{inl}]$$

$$\frac{\{x\} \vdash x : \varphi \;\rightarrow\; \vdash u_1[x] : \phi \quad \{y\} \vdash y : \psi \;\rightarrow\; \vdash u_2[y] : \phi \quad \vdash t : \varphi \vee \psi}{\vdash \mathbf{case}\, t\ \mathbf{of}\ (\mathbf{inl}\, x) \mapsto u_1[x] \mid (\mathbf{inr}\ y) \mapsto u_2[y] : \phi}\ [\&\mathrm{case}]$$

Lax logic

Primitive monad or lax modality [Curry, 1952]

$\Diamond\,\sqcup \quad : \quad type \;\longrightarrow\; type.$

$$\vdash unit : \varphi \Rightarrow \Diamond\varphi^{\,[\&\mathrm{unit}]}$$

$$\vdash bind : (\varphi \Rightarrow \Diamond\psi) \Rightarrow \Diamond\varphi \Rightarrow \Diamond\psi^{\,[\&\mathrm{bind}]}$$

Delimited control

Fixed answer type

$o \quad : \quad type.$

$$\vdash \mathbf{reset} : \Diamond\varphi \Rightarrow \Diamond\varphi \;^{[\&\mathrm{reset}]}$$

$$\vdash \mathbf{shift} : ((\varphi \Rightarrow \Diamond o) \Rightarrow \Diamond o) \Rightarrow \varphi \;^{[\&\mathrm{shift}]}$$

Monadic reflection

Reflect/reify are definable from **shift**/**reset**:

$$\begin{array}{rcl} [t] & = & \textbf{reset}\ (\mathit{unit}\ t). \\ \mu(t) & = & \textbf{shift}\ (\lambda k{:}\,\varphi \Rightarrow \Diamond o.\mathit{bind}\ k\ t). \end{array}$$

Lemma. *The following typing rules are derivable:*

$$\frac{\vdash t : \varphi}{\vdash [t] : \Diamond\varphi}\ [\&\text{reify}] \qquad \frac{\vdash t : \Diamond\varphi}{\vdash \mu(t) : \varphi}\ [\&\text{reflect}]$$

%solve $\vdash t : \varphi \ \rightarrow\ \vdash\ [t] : \Diamond\varphi$
%solve $\vdash t : \Diamond\varphi \ \rightarrow\ \vdash\ \mu(t) : \varphi$

Polymorphic monadic reflection

Polymorphic type for **shift**:

$$\vdash \textbf{shift} : \forall\alpha.((\varphi \Rightarrow \Diamond\alpha) \Rightarrow \Diamond\alpha) \Rightarrow \varphi \;^{[\&\text{shift}]}$$

Reflect is still definable from **shift**:

$$\mu(t) \quad = \quad \textbf{shift}\ (\Lambda\alpha.\lambda k{:}\, \varphi \Rightarrow \Diamond\alpha.\mathit{bind}\ k\ t).$$

Lemma. *The following typing rule is derivable:*

$$\frac{\vdash t : \Diamond\varphi}{\vdash \mu(t) : \varphi}\ [\&\text{reflect}]$$

%solve $\vdash t : \Diamond\varphi \;\longrightarrow\; \vdash \mu(t) : \varphi$

Remark. $\forall\alpha.((\varphi \Rightarrow \Diamond\alpha) \Rightarrow \Diamond\alpha)$ is equivalent to $\Diamond\varphi$.

%solve $\vdash t : \Diamond\varphi \;\longrightarrow\; \vdash \Lambda\alpha.\lambda k{:}\, \varphi \Rightarrow \Diamond\alpha.\mathit{bind}\ k\ t : \forall\alpha.((\varphi \Rightarrow \Diamond\alpha) \Rightarrow \Diamond\alpha)$
%solve $\vdash f : \forall\alpha.((\varphi \Rightarrow \Diamond\alpha) \Rightarrow \Diamond\alpha) \;\longrightarrow\; \vdash (f\,\{\varphi\}\ \mathit{unit}) : \Diamond\varphi$

Derived typing rule for reify

%lemma $\vdash t : \varphi \quad \Longrightarrow \quad \vdash \mathtt{[}t\mathtt{]} : \Diamond\varphi$

Proof.

$$\frac{\mathcal{D}_{\mathrm{of}_1}}{\vdash t : \varphi} \quad \Longrightarrow \quad \dfrac{\dfrac{}{\vdash \mathbf{reset} : \Diamond\varphi \Rightarrow \Diamond\varphi}[\&\mathrm{reset}] \qquad \dfrac{\dfrac{}{\vdash \mathit{unit} : \varphi \Rightarrow \Diamond\varphi}[\&\mathrm{unit}] \quad \dfrac{\mathcal{D}_{\mathrm{of}_1}}{\vdash t : \varphi}}{\vdash \mathit{unit}\ t : \Diamond\varphi}[\&\mathrm{app}]}{\vdash \mathbf{reset}\ (\mathit{unit}\ t) : \Diamond\varphi}[\&\mathrm{app}]\ ^{[\&]}$$

%mode $+\mathcal{D}_{\mathrm{of}_1} \quad \Longrightarrow \quad -\mathcal{D}_{\mathrm{of}_2}$
%worlds $() \quad \mathcal{D}_{\mathrm{of}_1} \quad \Longrightarrow \quad \mathcal{D}_{\mathrm{of}_2}$
%total $\{\} \quad \mathcal{D}_{\mathrm{of}_1} \quad \Longrightarrow \quad \mathcal{D}_{\mathrm{of}_2}$

Derived typing rule for reflect

%lemma $\vdash t : \Diamond\varphi \quad \Longrightarrow \quad \vdash \mu(t) : \varphi$

Proof.

$$\dfrac{\mathcal{D}_{\mathrm{of}_t}}{\vdash t : \Diamond\varphi} \quad \Longrightarrow \quad [\&\mathrm{shift}]\,\dfrac{[\alpha]\,\dfrac{\dfrac{[k]\,\dfrac{[\mathcal{D}_{\mathrm{of}_k}]\,\dfrac{[\&\mathrm{bind}]\,\dfrac{\dfrac{\mathcal{D}_{\mathrm{of}_k}}{\vdash k : \varphi \Rightarrow \Diamond\alpha}}{\vdash \mathit{bind}\ k : \Diamond\varphi \Rightarrow \Diamond\alpha}\,[\&\mathrm{app}] \qquad \dfrac{\mathcal{D}_{\mathrm{of}_t}}{\vdash t : \Diamond\varphi}}{\vdash \mathit{bind}\ k\ t : \Diamond\alpha}\,[\&\mathrm{app}]}{\vdash k : \varphi \Rightarrow \Diamond\alpha \ \longrightarrow\ \vdash \mathit{bind}\ k\ t : \Diamond\alpha}}{\{k\}\ \vdash k : \varphi \Rightarrow \Diamond\alpha \ \longrightarrow\ \vdash \mathit{bind}\ k\ t : \Diamond\alpha}}{\vdash \lambda k{:}\, \varphi \Rightarrow \Diamond\alpha.\mathit{bind}\ k\ t : (\varphi \Rightarrow \Diamond\alpha) \Rightarrow \Diamond\alpha}\,[\&\mathrm{lam}]}{\dfrac{\vdash \Lambda\alpha.\lambda k{:}\, \varphi \Rightarrow \Diamond\alpha.\mathit{bind}\ k\ t : \forall\alpha.((\varphi \Rightarrow \Diamond\alpha) \Rightarrow \Diamond\alpha)}{\vdash \mathbf{shift}\ (\Lambda\alpha.\lambda k{:}\, \varphi \Rightarrow \Diamond\alpha.\mathit{bind}\ k\ t) : \varphi}\,[\&\mathrm{app}]\ [\&]}\,[\&\mathrm{abs}]$$

%mode $+\mathcal{D}_{\mathrm{of}_1} \quad \Longrightarrow \quad -\mathcal{D}_{\mathrm{of}_2}$
%worlds $() \quad \mathcal{D}_{\mathrm{of}_1} \quad \Longrightarrow \quad \mathcal{D}_{\mathrm{of}_2}$
%total $\{\} \quad \mathcal{D}_{\mathrm{of}_1} \quad \Longrightarrow \quad \mathcal{D}_{\mathrm{of}_2}$

Different continuation monads

1. Continuation monad
   $\nabla\varphi \quad = \quad (\varphi \Rightarrow o) \Rightarrow o.$

2. Modal continuation monad
   $\nabla\varphi \quad = \quad (\varphi \Rightarrow \Diamond o) \Rightarrow \Diamond o.$

3. Polymorphic continuation monad
   $\nabla\varphi \quad = \quad \forall\alpha.(\varphi \Rightarrow \alpha) \Rightarrow \alpha.$

4. Polymorphic modal continuation monad
   $\nabla\varphi \quad = \quad \forall\alpha.(\varphi \Rightarrow \Diamond\alpha) \Rightarrow \Diamond\alpha.$

Remark. Cases 1 and 3 are obtained by taking $\Diamond$ as the identity monad.

Modal continuation monad

$$\nabla\varphi \quad = \quad (\varphi \Rightarrow \Diamond o) \Rightarrow \Diamond o.$$

$$\begin{aligned} unit_{\nabla} \quad &= \quad \lambda t\colon \varphi.\lambda k\colon \varphi \Rightarrow \Diamond o.(k\ t). \\ bind_{\nabla} \quad &= \quad \lambda k\colon \varphi \Rightarrow \nabla\psi.\lambda m\colon \nabla\varphi.\lambda c\colon \psi \Rightarrow \Diamond o.m\ (\lambda v\colon \varphi.k\ v\ c). \end{aligned}$$

%solve $\vdash unit_{\nabla} : \varphi \Rightarrow \nabla\varphi$

%solve $\vdash bind_{\nabla} : (\varphi \Rightarrow \nabla\psi) \Rightarrow \nabla\varphi \Rightarrow \nabla\psi$

Polymorphic modal continuation monad

$$\nabla\varphi \quad = \quad \forall\alpha.(\varphi \Rightarrow \Diamond\alpha) \Rightarrow \Diamond\alpha.$$

$$unit_{\nabla} \quad = \quad \lambda t{:}\,\varphi.\Lambda\alpha.\lambda k{:}\,\varphi \Rightarrow \Diamond\alpha.(k\ t).$$
$$bind_{\nabla} \quad = \quad \lambda m{:}\,\nabla\varphi.\lambda k{:}\,\varphi \Rightarrow \nabla\psi.\Lambda\alpha.\lambda c{:}\,\psi \Rightarrow \Diamond\alpha.m\{\alpha\}\ (\lambda v{:}\,\varphi.(k\ v)\{\alpha\}\ c).$$

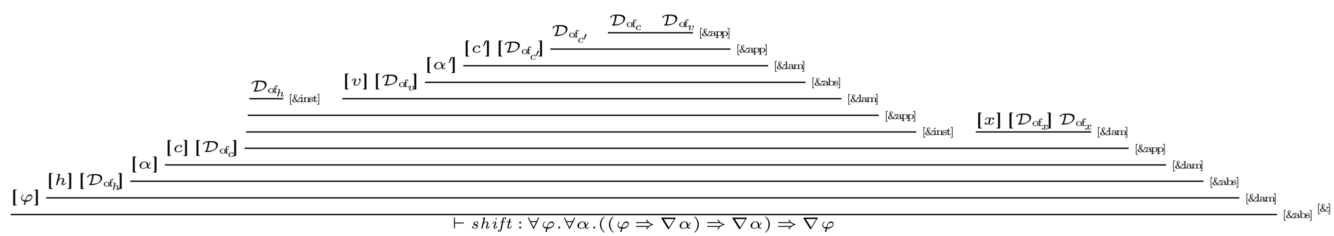
%solve $\vdash unit_{\nabla} : \varphi \Rightarrow \nabla\varphi$
%solve $\vdash bind_{\nabla} : \nabla\varphi \Rightarrow (\varphi \Rightarrow \nabla\psi) \Rightarrow \nabla\psi$

Polymorphic continuation monad (shift)

$$
\begin{aligned}
\nabla\varphi &= \forall\alpha.(\varphi \Rightarrow \alpha) \Rightarrow \alpha. \\
\mathit{shift} &= \Lambda\varphi.\lambda h{:}\,\forall\alpha.((\varphi \Rightarrow \nabla\alpha) \Rightarrow \nabla\alpha). \\
&\qquad \Lambda\alpha.\lambda c{:}\,\varphi \Rightarrow \alpha. \\
&\qquad\quad ((h\{\alpha\}\ (\lambda v{:}\,\varphi.\Lambda\alpha'.\lambda c'{:}\,\alpha \Rightarrow \alpha'.c'\ (c\ v)))\{\alpha\}\ \lambda x{:}\,\alpha.x).
\end{aligned}
$$

%lemma $\vdash \mathit{shift} : \forall\varphi.(\forall\alpha.((\varphi \Rightarrow \nabla\alpha) \Rightarrow \nabla\alpha)) \Rightarrow \nabla\varphi$

Proof.



%mode $-\mathcal{D}_{\mathrm{of}}$
%worlds $()$ $\mathcal{D}_{\mathrm{of}}$
%total $\{\}$ $\mathcal{D}_{\mathrm{of}}$

Polymorphic modal continuation monad (shift)

$$
\begin{array}{lcl}
\nabla\varphi & = & \forall\alpha.(\varphi \Rightarrow \Diamond\alpha) \Rightarrow \Diamond\alpha. \\
\mathit{shift} & = & \Lambda\varphi.\lambda h{:}\, \forall\alpha.((\varphi \Rightarrow \nabla\Diamond\alpha) \Rightarrow \nabla\Diamond\alpha). \\
 & & \quad \Lambda\alpha.\lambda c{:}\, \varphi \Rightarrow \Diamond\alpha. \\
 & & \quad\quad ((h\{\alpha\}\ (\lambda v{:}\, \varphi.\Lambda\alpha'.\lambda c'{:}\, \Diamond\alpha \Rightarrow \Diamond\alpha'.c'\ (c\ v)))\{\alpha\}\ \lambda x{:}\, \Diamond\alpha.x).
\end{array}
$$

%lemma $\vdash \mathit{shift} : \forall\varphi.(\forall\alpha.((\varphi \Rightarrow \nabla\Diamond\alpha) \Rightarrow \nabla\Diamond\alpha)) \Rightarrow \nabla\varphi$

Proof.

$$
\dfrac{[\varphi]\quad \dfrac{[h]\ [\mathcal{D}_{\mathrm{of}_h}]\quad \dfrac{[\alpha]\quad \dfrac{[c]\ [\mathcal{D}_{\mathrm{of}_c}]\quad \dfrac{\dfrac{\dfrac{\mathcal{D}_{\mathrm{of}_h}}{}\,[\&inst]\quad \dfrac{[v]\ [\mathcal{D}_{\mathrm{of}_v}]\quad \dfrac{[\alpha']\quad \dfrac{[c']\ [\mathcal{D}_{\mathrm{of}_{c'}}]\quad \dfrac{\mathcal{D}_{\mathrm{of}_{c'}}\quad \dfrac{\mathcal{D}_{\mathrm{of}_c}\quad \mathcal{D}_{\mathrm{of}_v}}{}\,[\&app]}{}\,[\&app]}{}\,[\&lam]}{}\,[\&abs]}{}\,[\&lam]}{}\,[\&app]}{}\,[\&inst]\quad \dfrac{[x]\ [\mathcal{D}_{\mathrm{of}_x}]\ \mathcal{D}_{\mathrm{of}_x}}{}\,[\&lam]}{}\,[\&app]}{}\,[\&lam]}{}\,[\&abs]}{}\,[\&lam]}{\vdash \mathit{shift} : \forall\varphi.\forall\alpha.((\varphi \Rightarrow \nabla\Diamond\alpha) \Rightarrow \nabla\Diamond\alpha) \Rightarrow \nabla\varphi}\,[\&abs]\ [\&]
$$

%mode $-\mathcal{D}_{\mathrm{of}}$
%worlds $()\quad \mathcal{D}_{\mathrm{of}}$
%total $\{\}\quad \mathcal{D}_{\mathrm{of}}$

Example: the continuation monad

$$\Diamond\varphi \quad = \quad (\varphi \Rightarrow o) \Rightarrow o.$$

$$unit_{\varphi} \quad = \quad \lambda t{:}\,\varphi.\lambda k{:}\,\varphi \Rightarrow o.(k\ t).$$
$$bind_{\varphi,\psi} \quad = \quad \lambda k{:}\,\varphi \Rightarrow \Diamond\psi.\lambda m{:}\,\Diamond\varphi.\lambda c{:}\,\psi \Rightarrow o.m\ (\lambda v{:}\,\varphi.k\ v\ c).$$

%solve $\vdash unit_{\varphi} : \varphi \Rightarrow \Diamond\varphi$
%solve $\vdash bind_{\varphi,\psi} : (\varphi \Rightarrow \Diamond\psi) \Rightarrow \Diamond\varphi \Rightarrow \Diamond\psi$

$$\vdash \mathbf{reset} : \Diamond\varphi \Rightarrow \Diamond\varphi\ ^{[\mathrm{reset}]}$$
$$\vdash \mathbf{shift} : \forall\alpha.((\varphi \Rightarrow \Diamond\alpha) \Rightarrow \Diamond\alpha) \Rightarrow \varphi\ ^{[\mathrm{shift}]}$$

$$[t] \quad = \quad \mathbf{reset}\ (unit_{\varphi}\ t).$$
$$\mu(t) \quad = \quad \mathbf{shift}\ (\Lambda\alpha.\lambda k{:}\,\varphi \Rightarrow \Diamond\alpha.bind_{\varphi,\alpha}\ k\ t).$$

%solve $\vdash t : \varphi \ \longrightarrow\ \vdash [t] : \Diamond\varphi$
%solve $\vdash t : \Diamond\varphi \ \longrightarrow\ \vdash \mu(t) : \varphi$

Example: the continuation monad

Defining **escape** in direct style:

$$\textbf{escape} \quad = \quad \lambda h\colon (\varphi \Rightarrow \psi) \Rightarrow \varphi . \mu(\lambda c\colon \varphi \Rightarrow o . \texttt{[} h \ \lambda a\colon \varphi . \mu(\lambda c'\colon \psi \Rightarrow o . c \ a) \texttt{]} \ c).$$

$$\textbf{\%solve} \quad \vdash \textbf{escape} : ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$$

What is the logical meaning of $\Diamond\varphi \Rightarrow \varphi$?

Since $\Diamond\varphi \equiv (\varphi \Rightarrow o) \Rightarrow o$, for some formula o, we get $\neg\neg\varphi \Rightarrow \varphi$ which extends the logic to classical logic if we take $o = \bot$, but this axiom is incoherent if o is a theorem (note that it is always at least classical logic).

Example: the state monad

$$\sigma \quad : \quad type.$$
$$\Diamond\varphi \quad = \quad \sigma \Rightarrow (\varphi \wedge \sigma).$$

$$unit_{\varphi} \quad = \quad \lambda a{:}\, \varphi.\lambda s{:}\, \sigma.\langle a, s\rangle.$$
$$bind_{\varphi,\psi} \quad = \quad \lambda f{:}\, \varphi \Rightarrow \Diamond\psi.\lambda t{:}\, \Diamond\varphi.\lambda s{:}\, \sigma.\mathbf{let}\ \langle x, s'\rangle = t\ s\ \mathbf{in}\ f\ x\ s'.$$

%solve $\vdash unit_{\varphi} : \varphi \Rightarrow \Diamond\varphi$
%solve $\vdash bind_{\varphi,\psi} : (\varphi \Rightarrow \Diamond\psi) \Rightarrow \Diamond\varphi \Rightarrow \Diamond\psi$

$$\vdash \mathbf{reset} : \Diamond\varphi \Rightarrow \Diamond\varphi\ ^{[\mathrm{reset}]}$$
$$\vdash \mathbf{shift} : \forall\alpha.((\varphi \Rightarrow \Diamond\alpha) \Rightarrow \Diamond\alpha) \Rightarrow \varphi\ ^{[\mathrm{shift}]}$$

$$[t] \quad = \quad \mathbf{reset}\ (unit_{\varphi}\ t).$$
$$\mu(t) \quad = \quad \mathbf{shift}\ (\Lambda\alpha.\lambda k{:}\, \varphi \Rightarrow \Diamond\alpha.bind_{\varphi,\alpha}\ k\ t).$$

%solve $\vdash t : \varphi \ \rightarrow\ \vdash [t] : \Diamond\varphi$
%solve $\vdash t : \Diamond\varphi \ \rightarrow\ \vdash \mu(t) : \varphi$

Example: the state monad

Defining **fetch** and **store** in direct style:

$$\begin{aligned}\textbf{unit} &= \forall\alpha.\alpha \Rightarrow \alpha.\\ \langle\rangle &= \Lambda\alpha.\lambda x{:}\,\alpha.x.\end{aligned}$$

$$\textbf{\%solve} \quad \vdash \langle\rangle : \textbf{unit}$$

$$\begin{aligned}\textbf{store} &= \lambda n{:}\,\sigma.\mu(\lambda s{:}\,\sigma.\langle\langle\rangle, n\rangle).\\ \textbf{fetch} &= \lambda x{:}\,\textbf{unit}.\mu(\lambda s{:}\,\sigma.\langle s, s\rangle).\end{aligned}$$

$$\begin{aligned}&\textbf{\%solve} \quad \vdash \textbf{fetch} : \textbf{unit} \Rightarrow \sigma\\ &\textbf{\%solve} \quad \vdash \textbf{store} : \sigma \Rightarrow \textbf{unit}\end{aligned}$$

What is the logical meaning of $\Diamond\varphi \Rightarrow \varphi$?

Since $\Diamond\varphi \equiv \sigma \Rightarrow (\varphi \wedge \sigma)$, for some formula σ, we get $(\sigma \Rightarrow (\varphi \wedge \sigma)) \Rightarrow \varphi$ which is not valid in general. This axiom is derivable if σ is a theorem, but it is incoherent if we take $\sigma = \bot$.

Example: the exception monad

$$\begin{aligned} \varepsilon &: type. \\ \Diamond\varphi &= \varphi \vee \varepsilon. \end{aligned}$$

$$\begin{aligned} unit_{\varphi} &= \lambda a{:}\,\varphi.\mathbf{inl}\,a. \\ bind_{\varphi,\psi} &= \lambda f{:}\,\varphi \Rightarrow \Diamond\psi.\lambda t{:}\,\Diamond\varphi.\mathbf{case}\,t\,\mathbf{of}\ (\mathbf{inl}\,a) \mapsto f\ a \mid (\mathbf{inr}\,b) \mapsto \mathbf{inr}\,b. \end{aligned}$$

$$\begin{aligned} &\mathbf{\%solve} \quad \vdash unit_{\varphi} : \varphi \Rightarrow \Diamond\varphi \\ &\mathbf{\%solve} \quad \vdash bind_{\varphi,\psi} : (\varphi \Rightarrow \Diamond\psi) \Rightarrow \Diamond\varphi \Rightarrow \Diamond\psi \end{aligned}$$

$$\vdash \mathbf{reset} : \Diamond\varphi \Rightarrow \Diamond\varphi\ ^{[\mathrm{reset}]}$$

$$\vdash \mathbf{shift} : \forall\alpha.((\varphi \Rightarrow \Diamond\alpha) \Rightarrow \Diamond\alpha) \Rightarrow \varphi\ ^{[\mathrm{shift}]}$$

$$\begin{aligned} [t] &= \mathbf{reset}\ (unit_{\varphi}\ t). \\ \mu(t) &= \mathbf{shift}\ (\Lambda\alpha.\lambda k{:}\,\varphi \Rightarrow \Diamond\alpha.bind_{\varphi,\alpha}\ k\ t). \end{aligned}$$

$$\begin{aligned} &\mathbf{\%solve} \quad \vdash t : \varphi \ \longrightarrow\ \vdash [t] : \Diamond\varphi \\ &\mathbf{\%solve} \quad \vdash t : \Diamond\varphi \ \longrightarrow\ \vdash \mu(t) : \varphi \end{aligned}$$

Example: the exception monad

Defining **raise** and **handle** in direct style:

$$\begin{array}{lcl} \mathbf{raise} & = & \lambda e{:}\,\varepsilon.\mu(\mathbf{inr}\, e). \\ \mathbf{handle} & = & \lambda t{:}\,\varphi.\lambda h{:}\,\varepsilon \Rightarrow \varphi.\mathbf{case}\ [t]\ \mathbf{of}\ (\mathbf{inl}\, a) \mapsto a \mid (\mathbf{inr}\, e) \mapsto h\ e. \end{array}$$

$$\begin{array}{ll} \mathbf{\%solve} & \vdash \mathbf{raise} : \varepsilon \Rightarrow \alpha \\ \mathbf{\%solve} & \vdash \mathbf{handle} : \varphi \Rightarrow (\varepsilon \Rightarrow \varphi) \Rightarrow \varphi \end{array}$$

What is the logical meaning of $\Diamond\varphi \Rightarrow \varphi$?

Since $\Diamond\varphi \equiv \varphi \vee \varepsilon$, for some formula ε, we get $(\varphi \vee \varepsilon) \Rightarrow \varphi$ which is not valid in general. This axiom is incoherent if ε is a theorem, but it is derivable if $\neg\varepsilon$ is derivable.

Concluding remarks

- Depending on $\diamondsuit$, the type of **shift** can be:
  - *intuitionistic*
  - *classical*
  - *incoherent*
- However a proof of $\vdash \perp$ is translated into a proof $\vdash \diamondsuit \perp$ (and the target logic is consistent since $\diamondsuit$ is defined)
- In a dependently typed framework $\vdash \mathbf{reset} : \diamondsuit\varphi \Rightarrow \diamondsuit\varphi$ is useless. Is it just an optimization?
- Similarly, we can replace **shift** by the $\mathcal{D}$-operator.
- Why should we expect **shift** to be logically sound?

Bibliography

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