1 Simply typed λ-calculus

1.1 Syntax

1.1.1 Types

$type \quad : \quad \mathbf{type}.$

$\mathbf{nat} \quad : \quad type.$
$\sqcup \rightarrow \sqcup \quad : \quad type \rightarrow type \rightarrow type.$

1.1.2 Terms

$term \quad : \quad \mathbf{type}.$

$\mathbf{0} \quad : \quad term.$
$\mathbf{rec} \quad : \quad term.$
$\sqcup \quad : \quad ident \rightarrow term.$
$\sqcup\ \sqcup \quad : \quad term \rightarrow term \rightarrow term.$
$\lambda \sqcup : \sqcup . \sqcup \quad : \quad ident \rightarrow type \rightarrow term \rightarrow term.$

$\mathbf{let}\ x{:}\,\tau = u\ \mathbf{in}\ t \quad = \quad (\lambda x{:}\,\tau . t)\, u.$
$\mathbf{rec}(t_1, t_2, t_3) \quad = \quad (\mathbf{rec}\, t_1\, t_2\, t_3).$

1.1.3 Contexts

$cxt \quad : \quad \mathbf{type}.$

$\{\} \quad : \quad cxt.$
$\sqcup , \sqcup : \sqcup \quad : \quad cxt \rightarrow ident \rightarrow type \rightarrow cxt.$

1.2 Lookup

$\_ : \_ \in \_ \quad : \quad \mathit{ident} \;\rightarrow\; \mathit{type} \;\rightarrow\; \mathit{cxt} \;\rightarrow\; \textbf{type}.$

$$\frac{}{x\colon \tau \in \Gamma, x\colon \tau}\,[\text{lookup\_1}]$$

$$\frac{x\colon \tau \in \Gamma \quad x \neq x'}{x\colon \tau \in \Gamma, x'\colon \tau'}\,[\text{lookup\_2}]$$

1.3 Typing judgment

$\_ \vdash \_ : \_ \quad : \quad \mathit{cxt} \;\rightarrow\; \mathit{term} \;\rightarrow\; \mathit{type} \;\rightarrow\; \textbf{type}.$

$$\Gamma \vdash \mathbf{0}\colon \mathbf{nat}\,^{[\text{of\_zero}]}$$

$$\Gamma \vdash \mathbf{rec}\colon \mathbf{nat} \to \tau \to (\mathbf{nat} \to \tau \to \tau) \to \tau\,^{[\text{of\_rec}]}$$

$$\frac{x\colon \tau \in \Gamma}{\Gamma \vdash x\colon \tau}\,[\text{of\_var}]$$

$$\frac{\Gamma, x\colon \tau \vdash t\colon \tau'}{\Gamma \vdash \lambda x\colon \tau.t\colon \tau \to \tau'}\,[\text{of\_lam}]$$

$$\frac{\Gamma \vdash t_1\colon \tau \to \tau' \quad \Gamma \vdash t_2\colon \tau}{\Gamma \vdash t_1\, t_2\colon \tau'}\,[\text{of\_app}]$$

1.3.1 Derived rule for the rec macro

$$\Gamma \vdash t_1 \colon \mathbf{nat} \quad \bigwedge \quad \Gamma \vdash t_2 \colon \tau \quad \bigwedge \quad \Gamma \vdash t_3 \colon (\mathbf{nat} \to \tau \to \tau) \quad \Longrightarrow \quad \Gamma \vdash \mathbf{rec}(t_1, t_2, t_3) \colon \tau \quad : \quad \mathbf{type}.$$

$$\dfrac{\mathcal{D}_{\mathrm{of}_1}}{\Gamma \vdash t_1 \colon \mathbf{nat}} \quad \bigwedge \quad \dfrac{\mathcal{D}_{\mathrm{of}_2}}{\Gamma \vdash t_2 \colon \tau} \quad \bigwedge \quad \dfrac{\mathcal{D}_{\mathrm{of}_3}}{\Gamma \vdash t_3 \colon (\mathbf{nat} \to \tau \to \tau)} \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \mathbf{rec} \colon \mathbf{nat} \to \tau \to (\mathbf{nat} \to \tau \to \tau) \to \tau}\,{\scriptstyle [\mathrm{of\_rec}]} \quad \dfrac{\mathcal{D}_{\mathrm{of}_1}}{\Gamma \vdash t_1 \colon \mathbf{nat}}}{\Gamma \vdash \mathbf{rec}\, t_1 \colon \tau \to (\mathbf{nat} \to \tau \to \tau) \to \tau}\,{\scriptstyle [\mathrm{of\_app}]} \quad \dfrac{\mathcal{D}_{\mathrm{of}_2}}{\Gamma \vdash t_2 \colon \tau}}{\Gamma \vdash \mathbf{rec}\, t_1\, t_2 \colon (\mathbf{nat} \to \tau \to \tau) \to \tau}\,{\scriptstyle [\mathrm{of\_app}]} \quad \dfrac{\mathcal{D}_{\mathrm{of}_3}}{\Gamma \vdash t_3 \colon (\mathbf{nat} \to \tau \to \tau)}}{\Gamma \vdash \mathbf{rec}(t_1, t_2, t_3) \colon \tau}\,{\scriptstyle [\mathrm{o}}$$

$$\begin{array}{l}
\mathbf{\%mode} \quad +\mathcal{D}_{\mathrm{of}_1} \quad \bigwedge \quad +\mathcal{D}_{\mathrm{of}_2} \quad \bigwedge \quad +\mathcal{D}_{\mathrm{of}_3} \quad \Longrightarrow \quad -\mathcal{D}_{\mathrm{rec}} \\
\mathbf{\%worlds} \quad () \quad \mathcal{D}_{\mathrm{of}_1} \quad \bigwedge \quad \mathcal{D}_{\mathrm{of}_2} \quad \bigwedge \quad \mathcal{D}_{\mathrm{of}_3} \quad \Longrightarrow \quad \mathcal{D}_{\mathrm{rec}} \\
\mathbf{\%total} \quad \{\} \quad \mathcal{D}_{\mathrm{of}_1} \quad \bigwedge \quad \mathcal{D}_{\mathrm{of}_2} \quad \bigwedge \quad \mathcal{D}_{\mathrm{of}_3} \quad \Longrightarrow \quad \mathcal{D}_{\mathrm{rec}}
\end{array}$$